Asymmetric Rényi Problem and PATRICIA Tries

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Rényi's Problem

We have a set X of objects and a set A of labels, along with a **hidden** bijective labeling $\phi : X \to A$.

Algorithmic task: Recover the labeling ϕ using as few queries as possible.

A **query** take the form of a subset B of labels. An answer to the query B is the set of objects with labels in B: $\phi^{-1}(B)$.

Rényi's problem in a nutshell: How many **random queries** are needed to recover ϕ in its entirety?

Object: x	Label: $\phi(x)$
1	d
2	e
3	a
4	c
5	b

Query: B	Response: $\phi^{-1}(B)$
{ a, c, e}	{2, 3, 4 }
{ d }	{ 1 }
{ a, b, c, d }	{ 1, 3, 4, 5 }

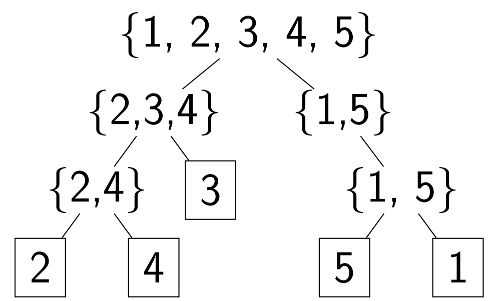
Partition Refinement Tree View of the Rényi Process

A sequence of queries corresponds to a refinement of partitions of the item set:

Example:

$\phi : 1 \to d, 2 \to e, 3 \to a, 4 \to c, 5 \to b$.

1. $B_1 = \{b, d\} \mapsto \{1, 5\}$
2. $B_2 = \{a, b, d\} \mapsto \{1, 3, 5\}$,
3. $B_3 = \{a, c, d\} \mapsto \{1, 3, 4\}$,



Level $j \geq 0$ of the tree $\iff$ partition $\mathcal{P}_j$.

Right child node $\iff$ subset of objects in parent set contained in the response to the jth query.

Singletons are only explicitly depicted in the first level in which they appear.

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$$\mathcal{P}_j = \{\{2,3,4\},\{1,5\}\}, \qquad \phi^{-1}(B_j) = \{1,3,5\}, \quad \text{or} \qquad \phi^{-1}(B_j) = \{2,4\}.$$

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- To construct $B_{j,i+1}$ from $B_{j,i}$: for each label in each partition element unsplit by $B_{j,i}$, decide again whether or not to include it independently with probability p: e.g., $\phi^{-1}(B_{j,1}) = \{3\}, \phi^{-1}(B_{j,2}) = \{3,5\}$.

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- Query B_j is the end result, where all partition elements are split: e.g., $\phi^{-1}(B_j) = \{3,5\}$.

Refining Inconclusive Queries: PATRICIA Trie Correspondence

Important correspondence: Partition refinement tree for the Rényi process with inconclusive query refinement $\stackrel{D}{=}$ a PATRICIA trie on n infinite random binary strings.

An object $\iff$ a string, where 1 means that the label is in a query, and 0 means that it is not.

Example: $\phi : 1 \to d, 2 \to e, 3 \to a, 4 \to c, 5 \to b$.

Strings corresponding to objects:

1. d: $111\ldots$
2. e: $000\ldots$
3. a: $011\ldots$
4. c: $0001\ldots$
5. b: $110\ldots$

Queries corresponding to strings:

1. $B_1 = \{b, d\} \mapsto \{1, 5\}$
2. $B_2' = \{a, b, d\} \mapsto \{1, 3, 5\}$;
   $B_2 = \{a, d\} \mapsto \{1, 3\}$,
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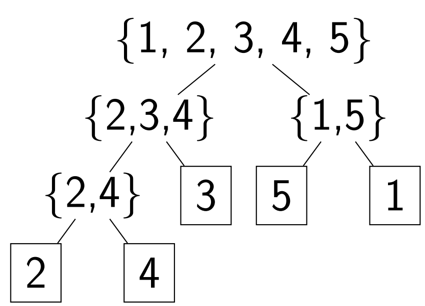
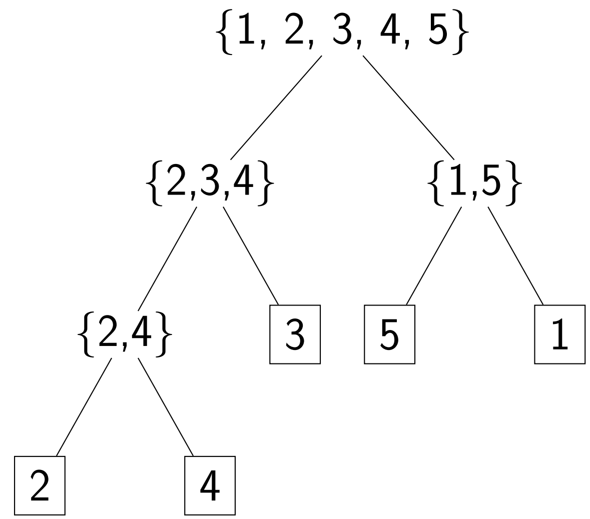


Parameters of Interest

Parameters of interest:

- Height (H_n): # of queries needed to recover ϕ entirely.
- Fillup level (F_n): # of queries needed before the first item-label pair is discovered.
- Typical depth (D_n): # of queries before a randomly chosen item's label is discovered.
- External profile at level k $(B_{n,k})$: Number of item-label pairs revealed by the kth query.



In the diagram: $H_5 = 3$, $F_5 = 2$, $\Pr[D_5 = 2] = 3/5, \Pr[D_5 = 3] = 2/5$, $B_{5,2} = 3$, $B_{5,3} = 2$.

Our Results

We have the following asymptotic expansions for the typical values of H_n and F_n:

Theorem 1 (Asymptotics for F_n and H_n)**.** *With high probability,*

$$H_n = \begin{cases} \log_{1/p} n + \frac{1}{2}\log_{p/q}\log n + o(\log\log n) & p > q = 1-p \\ \log_2 n + \sqrt{2\log_2 n} + o(\sqrt{\log n}) & p = q = 1/2 \end{cases} \tag{1}$$

and

$$F_n = \begin{cases} \log_{1/q} n - \log_{1/q}\log\log n + o(\log\log\log n) & p > q = 1-p \\ \log_2 n - \log_2\log n + o(\log\log n) & p = q = 1/2 \end{cases} \tag{2}$$

for large n*.*

Symmetric case ($p = 1/2$) was known, but **asymmetric case (**$p > 1/2$**) is new!**

Note the phase transition in the second term.

We also have **results for** D_n **via the external profile.**

Prior Work

Pittel & Rubin (1990): "How many random questions are needed to identify n distinct objects?": Two-term asymptotics for H_n in the symmetric case ($p = 1/2$) via (different) GF methods.

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Magner Ph.D. thesis / Magner & Szpankowski (2015): "Profiles of PATRICIA tries": Precisely analyzed distribution of the external profile in the central range.

This work: Requires **extension** of the external profile analysis to the **boundaries of the central range**.

Comparison with Tries and DSTs

Only the first terms of F_n and H_n for tries and DSTs with $p > q$ are given in the literature!

For tries:

$$H_n \sim \frac{2}{\log(1/(p^2+q^2))} \log n, \qquad F_n \sim \log_{1/q} n.$$

For DSTs:

$$H_n \sim \log_{1/p} n, \qquad F_n \sim \log_{1/q} n.$$

Proof Sketch: Derivation of Height

Goal: Define $k_* = k_*(n) = \log_{1/p} n + \psi(n)$. We want to determine $\psi(n)$ for which

$$H_n = k_* + o(\psi(n)).$$

First, we connect H_n with $B_{n,k}$: $H_n = \max\{k \ : \ B_{n,k} > 0\}$.

Then connect H_n to moments of $B_{n,k}$ via the first and second moment methods:

$$\Pr[H_n > k] \leq \sum_{j>k} \mathbb{E}[B_{n,j}] \qquad\qquad \Pr[H_n < k] \leq \frac{\mathbf{Var}[B_{n,k}]}{\mathbb{E}[B_{n,k}]^2}.$$

So we want $\psi(n)$ to satisfy

$$\mathbb{E}[B_{n,\log_{1/p} n+(1-\epsilon)\psi(n)}] \xrightarrow{n\to\infty} \infty, \qquad\qquad \mathbb{E}[B_{n,\log_{1/p} n+(1+\epsilon)\psi(n)}] \xrightarrow{n\to\infty} 0.$$

Derivation of the Height: External Profile Analysis

Ranges of behavior of $\mathbb{E}[B_{n,k}]$:

- Magner/Magner & Szpankowski (2015): Central range.

$$k \sim \alpha \log n, \qquad \alpha \in \left(\frac{1}{\log(1/q)} + \epsilon, \frac{1}{\log(1/p)} - \epsilon \right).$$

- **This work:** Boundaries of the central range.

$$H_n : k \sim \log_{1/p} n, \qquad F_n : k \sim \log_{1/q} n.$$

$$k = \log_{1/q} n \qquad\qquad k = \log_{1/p} n$$

$$k = \log_{1/q} n - \log_{1/q} \log\log n \qquad\qquad k = \log_{1/p} n + \tfrac{1}{2} \log_{p/q} \log n$$

External Profile Analysis (continued)

Basic tool chain for profile analysis:

$$\boxed{\text{Poissonization}} \longrightarrow \boxed{\text{Mellin}} \longrightarrow \boxed{\text{Inverse Mellin}} \longrightarrow \boxed{\text{De-Poissonization}}$$

Poisson transform $\tilde{G}_k(z)$ for $\mathbb{E}[B_{n,k}]$:

$$\tilde{G}_k(z) = \tilde{G}_{k-1}(pz) + \tilde{G}_{k-1}(qz) + e^{-pz}(\tilde{G}_k(qz) - \tilde{G}_{k-1}(qz)) + e^{-qz}(\tilde{G}_k(pz) - \tilde{G}_{k-1}(pz)).$$

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Explicit formula for Mellin transform of $\tilde{G}_k(z)$:

$$G_k^*(s) := \int_0^\infty z^{s-1}\tilde{G}_k(z)\,\mathrm{d}z = (p^{-s} + q^{-s})^k A_k(s)\Gamma(s+1).$$

Fundamental strip for $\tilde{G}_k(z)$: $\Re(s) > -k-1$.

Inverting the Mellin Transform

Main new challenge of our analysis: estimate $\tilde{G}_k(n)$ by bounding inverse Mellin transform of $G_k^*(s)$:

$$\tilde{G}_k(z) = \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} z^{-s} G_k^*(s) \, \mathrm{d}s = \int_{\rho - i\infty}^{\rho + i\infty} J_k(z, s) \, \mathrm{d}s.$$

where $\rho > -k - 1$ and

$$J_k(n, s) = \sum_{j=0}^{k} n^{-s} (p^{-s} + q^{-s})^{k-j} \sum_{m \geq j} (p^m + q^m)(\mathbb{E}[B_{m,j}] - \mathbb{E}[B_{m,j-1}]) \frac{\Gamma(m+s)}{\Gamma(m+1)}.$$

Main Steps of Upper Bounding the Inverse Mellin Integral

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- Exponential decay of $\Gamma(s+1)A_k(s)$ along vertical lines $\Longrightarrow$

$$|\tilde{G}_k(n)| = O(J_k(n,\rho))$$

for $\rho > -k-1$.

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  for $\rho > -k-1$.

- Estimate each jth term of $J_k(n,\rho)$ using upper bound on $\mathbb{E}[B_{m,j}]$ for j sufficiently close to m:

$$\mathbb{E}[B_{m,j}] \leq C\frac{m!}{(m-j-1)!}p^{j^2/2+j/2+o(j)}.$$

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- Maximize resulting upper bound over all j:

$$\boxed{J_k(n,s) \leq p^{\nu(n,s)},}$$

  where

$$\nu(n,s) = -\frac{(s+\log_{1/p}(1+(p/q)^s)+\psi(n)+1)^2}{2}$$
$$-\log_{1/p} n \log_{1/p}(1+(p/q)^s) + \psi(n)^2/2 + o(\psi(n)^2).$$

Tightening the Upper Bound on the Poisson Transform

We now know

$$\boxed{\tilde{G}_k(n) = O(J_k(n,s)) \le p^{\nu(n,s)}}$$

for $s \in (-k-1, 0)$.

Tighten the upper bound by minimizing over s:

- $p = q = 1/2$: $\log_{1/p}(1 + (p/q)^s) = 1$, and we get

$$s_* = -\psi(n) + O(1), \qquad \nu(n, s_*) = -\log_2 n + \psi(n)^2/2 + o(\psi(n)^2).$$

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- $p > q$: $\log_{1/p}(1 + (p/q)^s)$ is a function of s. Lambert W function asymptotics + algebra $\Longrightarrow$

$$s_* = -\log_{p/q} \log n + O(\log\log\log n).$$

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- So we need $\psi(n)$ to be such that $\nu(n, s_*) = 0$.

$$\psi(n) = \begin{cases} \sqrt{\log_2 n} + o(\sqrt{\log n}) & p = q = 1/2 \\ \frac{1}{2}\log_{p/q} \log n + O(\log\log\log n) & p > q. \end{cases}$$

Plugging in $k = \log_{1/p} n + (1 \pm \epsilon)\psi(n)$ for the upper bound on $\tilde{G}_k(n)$ gives

$$p^{\frac{\epsilon}{2}(\log_{p/q}\log n)^2 + o((\log\log n)^2)} \to 0, \qquad p^{-\frac{\epsilon}{2}(\log_{p/q}\log n)^2 + o((\log\log n)^2)} \to \infty,$$

as desired.

Possible Future Directions

- More precise asymptotics/limit laws for H_n and F_n. The limiting behavior for H_n is known for $p = q$ (Knessl & Szpankowski (1999): " Limit laws for height in generalized tries and PATRICIA tries"), but not for $p > q$.

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- More satisfying explanation of the phase transition for H_n in terms of the number and sizes of fringe subtrees.
- How does adding noise affect the process? One natural noise model: Items are dropped randomly from responses to queries, or extra items are added.

Thank you!