On some properties of characteristic function of limiting quicksort distribution

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Outline

Let X_n be the number of steps required by Quicksort algorithm to sort the list of values $\sigma(1), \sigma(2), \ldots, \sigma(n)$ where σ is a random permutation chosen with uniform probability from the set of all permutations S_n of order n.

Example

$$\overbrace{\boxed{8,4,3,2,1,3,5,7,6,9}}^{n}$$

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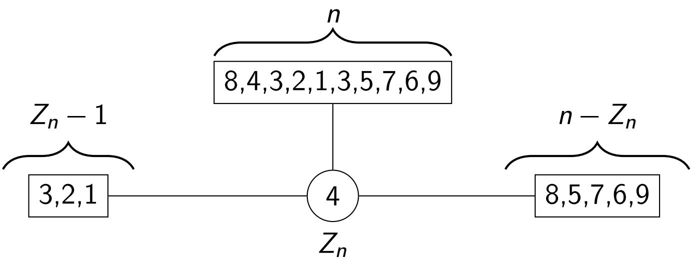
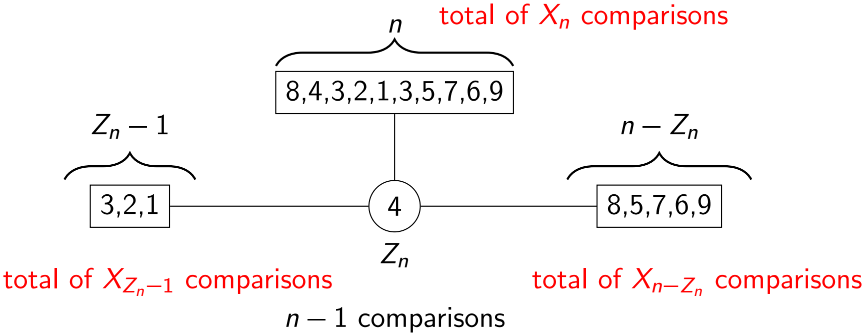


$n - 1$ comparisons

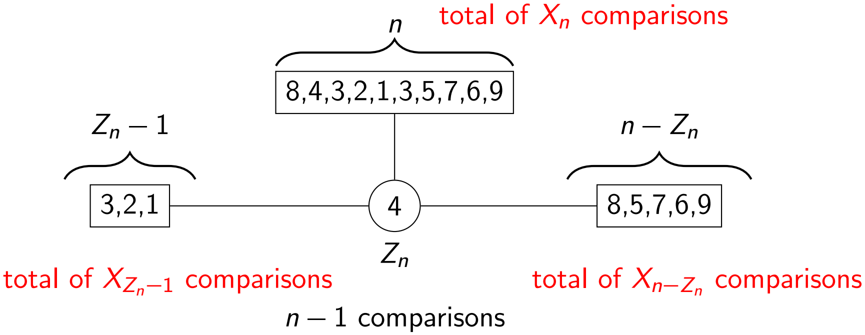
Example



n
total of X_n comparisons
8,4,3,2,1,3,5,7,6,9
$Z_n - 1$
3,2,1
4
Z_n
$n - Z_n$
8,5,7,6,9
total of $X_{Z_n - 1}$ comparisons
total of $X_{n - Z_n}$ comparisons
$n - 1$ comparisons

Example

$$X_n =^d X_{Z_n-1} + X'_{n-Z_n} + n - 1$$



The total number of comparisons X_n satisfies the recurent relation

$$X_n =^d X_{Z_n - 1} + X'_{n - Z_n} + n - 1, \quad X_0 = 0, X_1 = 0$$

where Z_n is uniformly distributed on the set $\{1, 2, 3, \ldots, n\}$.
Régnier (1989) and Rösler (1991) proved that X_n converges to some limit law

$$\frac{X_n - \mathbb{E}X_n}{n} \to^d Y$$

Let $f(t)$ be the characteristic function

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Knessl and Szpankowski (1999) using heuristic approach established a number of very precise estimates for the behavior of $p(x)$ at infinity.

Fill and Janson (2000) showed that that for all real $p > 0$ there is such a constant c_p that

$$|f(t)| \leqslant \frac{2^{p^2+6p}}{|t|^p}, \quad \text{for all} \quad t \in \mathbb{R}.$$

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The infimum in the above inequality can be evaluated as

$$|f(t)| \leqslant \inf_{p>0} \frac{2^{p^2+6p}}{|t|^p} \leqslant |t|^3 e^{-\frac{\log^2 |t|}{4 \log 2}}.$$

Theorem

There is a constant $\eta > 0$ *such that*

$$f(t) = O(e^{-\eta|t|})$$

as $|t| \to \infty$.

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Corollary

Quicksort distribution has a bounded density that can be extended analytically to the vicinity of the real line $|\Im(s)| < \eta$ *by means of formula*

$$p(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-ist}\, dt.$$

Where η *is the same positive number as in the formulation of Theorem.*

The main idea of the proof

Since

$$\frac{X_n - \mathbb{E}X_n}{n} \to^d Y$$

This yields (see Rösler (1991)) the functional equation

$$Y =^d Y\tau + Y'(1-\tau) + 2\tau\log\tau + 2(1-\tau)\log(1-\tau) + 1$$

where τ is independent of Y, Y' and is uniformly distributed on the interval $[0, 1]$.

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$$f(t) = e^{it} \int_0^1 f(tx) f\big(t(1-x)\big) e^{2itx \log x + 2it(1-x)\log(1-x)}\, dx$$

Hence the Laplace transform

$$\psi(s) = \int_0^\infty f(t) e^{2it\log t} e^{-st}\, dt$$

is analytic for all $\Re(s) > 0$ and satisfies the shift-differential equation

$$-\psi'(s) = \psi^2(s-i).$$

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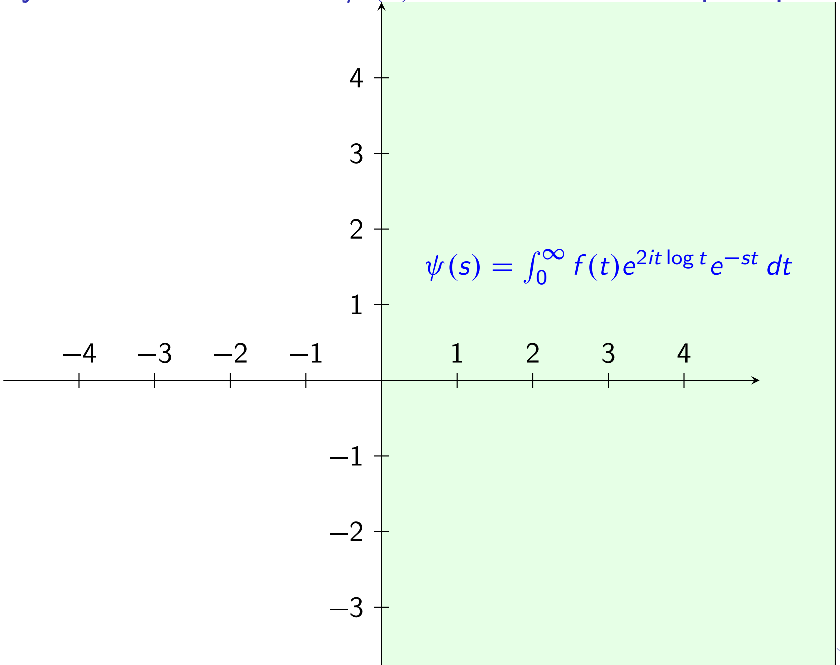
This equation leads to various upper bounds for the derivatives

$$\psi^{(j)}(s)$$

for the values of s located on the line $\Re s = 1$

Analytic continuation of $\psi(s)$ to the whole complex plane



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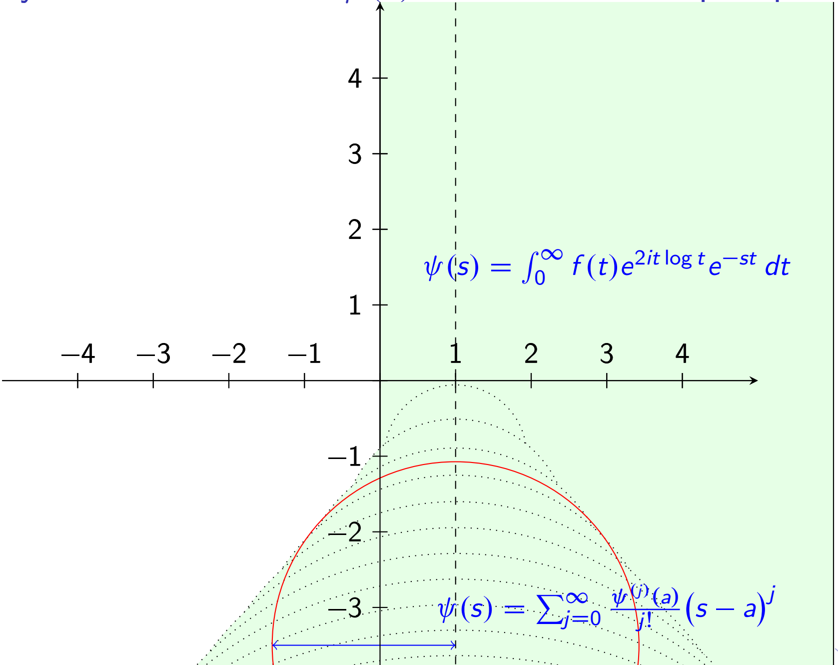

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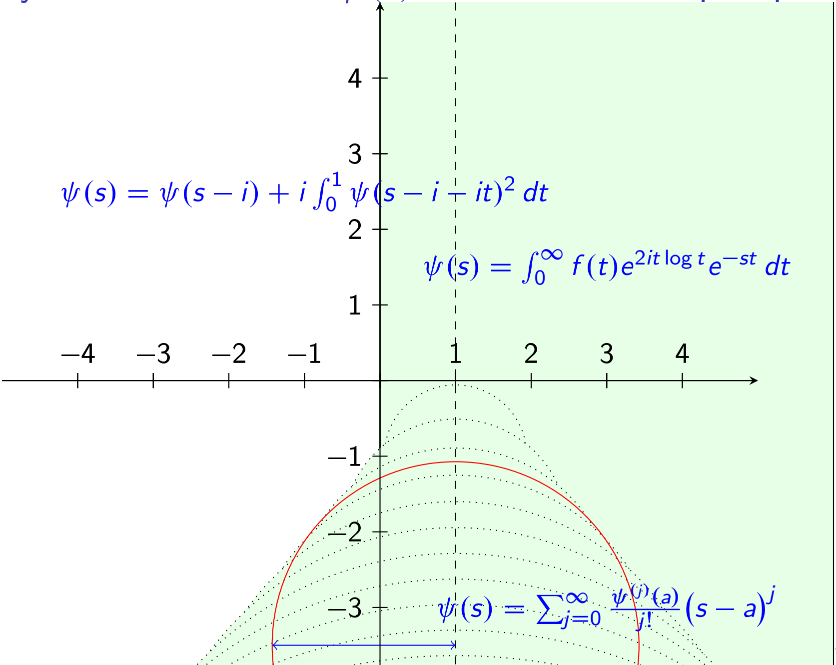

If we *formally* make change of variables $t = ui$ in the integral

$$\psi(s) = \int_0^\infty f(t)e^{2it\log t}e^{-st}\,dt$$

then

$$\psi(s) = \int_0^\infty f(iu)e^{-2\log u}e^{-(\pi+s)it}\,dt$$

Proposition

Function $\psi(s)$ satisfies the functional equation

$$-\overline{\psi(-\bar{s})} = \psi(s-2\pi).$$

This functional equation implies that $|\psi(s)|$ is symmetric with respect to the line $\Re s = -\pi$.

Corollary

$$\int_0^\infty |f(t)|e^{\pi t}\,dt = \infty$$

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