# On Green Leader Election

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# Leader election by Philippe Jaquet

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- You have n (unknown) competitors

- How to elect a leader
  - With equal chances

# Maximum propagation technique

Maximum propagation: technique propagated by **C. Baquero, P. S. Almeida, R. Menezes** (2009), *Fast estimation of aggregates in unstructured networks*

## Description

1. We have a linearly ordered set $S = \{1, \ldots, L\}$ or $S = \mathbb{N}$
2. each node $v \in \{1, \ldots, n\}$ selects independently a number $a_v \in S$ using the same distribution
3. winner - this node which selects $\max(\{a_v : v = 1, \ldots, n\})$

## Remarks

- PLUS: easy to implement in Ad Hoc network
- WARNING: we must control collisions

# MaxPropagation

## Probability of Success

$$\Pr[\text{max is unique}] = \binom{n}{1} \Pr[\text{"1" is the unique max}] =$$

$$n \sum_{k \in S} \Pr[\text{"1" select k}] \Pr[\text{"1" selects number } < k]^{n-1}$$

## Uniform distribution on $\{1, \ldots, L\}$

$$\Pr[\text{max is unique}] = n \sum_{k=1}^{L} \frac{1}{L} \left( \frac{k-1}{L} \right)^{n-1} =$$

$$\frac{n}{L^n} \sum_{k=0}^{L-1} k^{n-1} > 1 - \frac{n}{L}$$

# How to select a loser?

Put $S = \{0, 1, \ldots, 2^M - 1\}$ and $L = 2^M$ in previous formula:

**Leader Binary Election**

$$\Pr[\text{max is unique}] = \frac{n}{2^{Mn}} \sum_{k=0}^{2^M-1} k^{n-1} > 1 - \frac{n}{2^M}$$

- **H. Prodinger**, (1993), *How to select a loser*
- **J. A. Fill, H. Mahmoud, W. Szpankowski**, (1995), *On the Distribution for the Duration of a Randomized Leader Election Algorithm*

# Transformation of max-propagation to beep model

- represent $x \in \{0, \ldots, 2^M - 1\}$ as $x = \sum_{i=1}^{M} \epsilon_i 2^{M-i}$
- use $(\epsilon_1, \ldots, \epsilon_M)$ in the following game:

### Leader Binary Election

Leader = **true**
**for** $i = 1 \ldots M$ **do**
    **if** $\epsilon_i = 1$ **then**
        send beep
    **else**
        listen
        **if** beep or collision **then**
            Leader = **false**
            exit
        **end if**
    **end if**
**end for**

# Leader Green Election

**P. Jacquet, D. Milioris, P. Muhlethaler**, 2013, *A novel energy efficient broadcast leader election*

- each node $v$ selects independently a random number $X_v$ according to Geo(p) distribution
- $X_v = \epsilon_0 3^0 + \epsilon_1 3^1 + \ldots + \epsilon_M 3^M$, $\epsilon_i \in \{0, 1, 2\}$
- node $v$ uses the sequence

$$b(\epsilon_M) \| b(\epsilon_{M-1}) \| \ldots \| b(\epsilon_0)$$

  in the previous algorithm, where $b(0) = 00$, $b(1) = 01$ and $b(2) = 11$.
- winner: the node which selects max

FIRST QUESTION: how large $M$ should w choose ?

# Max Geo

### Definition

$X \sim \mathrm{MGeo}(n,p)$ iff there are independent $X_1, \ldots X_n \sim \mathrm{Geo}(p)$ and

$$X = \max\{X_1, \ldots, X_n\}$$

### Theorem

*If* $X \sim \mathrm{MGeo}(n,p)$ *then*

$$\Pr[X > \frac{C \ln n}{\ln \frac{1}{1-p}}] < \frac{1}{n^{C-1}}$$

### Example

If $p = \frac{1}{100}$ and $n \leq 10^5$ then $\Pr[X > 8200] < \frac{1}{10^{30}}$,
so we need

$$2 \cdot (\lceil \log_3(8200) \rceil + 1) = 20 \text{ bits.}$$

# Number of winers

**Definition**

$W \sim \mathrm{WMGeo}(n,p)$ iff there are independent $X_1, \ldots X_n \sim \mathrm{Geo}(p)$ and

$$W = \mathrm{card}(\{i : X_i = \max\{X_1, \ldots, X_n\}\})$$

- **P. Kirschenhofer and H. Prodinger** (1996), *The Number of Winners in a Discrete Geometrically Distributed Sample*
- **G. Louchard and H. Prodinger** (2009), *The asymmetric leader election algorithm: Another approach.*

Thanks to an anonymous reviewer !!!

# Number of winners

**Theorem**

*If* $W_{n,p} \sim \mathrm{WMGeo}(n,p)$ *then*

$$\Pr[W_{n,p} = a] = \binom{n}{a} p^a \sum_{b=0}^{n-a} \binom{n-a}{b} \frac{(-1)^b}{1-q^{a+b}}$$

*and*

$$\mathbf{E}\left[W_{n,p}\right] = \frac{np}{q} \sum_{b=0}^{n-1} \binom{n-1}{b} \frac{(-1)^b}{1-q^{b+1}} .$$

We may use Rice method for finding precise approximations of these formulas.

# Number of winers

**Theorem**

*If* $W \sim \mathrm{WMGeo}(n,p)$ *then* $\mathbf{E}[W] = \frac{1}{1-p}\Pr[W_{n,p} = 1]$ .

**Corollary**

*If* $W \sim \mathrm{WMGeo}(n,p)$ *then* $\Pr[W = 1] \geq 1 - \frac{1}{2}\frac{p}{1-\frac{p}{2}} \approx 1 - \frac{1}{2}p$

This is an "almost" precise result !!!. More precisely:

**Rice analysis**

$$\Pr[W = 1] \approx 1 - \frac{p}{2} - \frac{p^2}{12} - \frac{p^3}{24}$$

So: probability of failure is QUITE large.

# Number of winers: Rice analysis

Let us fix $p \in (0,1)$, let $Q = \frac{1}{1-p}$. We put

$$\phi_p(a) = \frac{p^a}{a \ln Q} + \frac{(a+1)^2}{12a} p^a \ln Q .$$

**Theorem**

*If* $W \sim \mathrm{WMGeo}(n,p)$ *then*

$$\Pr[W \geq k] < \frac{\phi(k)}{1-2p}$$

**Corollary**

*If* $W \sim \mathrm{WMGeo}(n,\frac{1}{100})$ *then*

$$\Pr[W > 10] \leq \frac{1}{10^{21}} .$$

# How to use Leader Green Election

For $n \leq 10^5$ and probability of success $\geq 1 - \frac{1}{10^{20}}$

LGE
20 bits

LBE
70 bits

Arbitrary $n$ and probability of success $\geq 1 - \frac{1}{10^{20}}$

$2\log_3\left(\frac{7\ln(n)}{\ln\frac{1}{1-0.01}}\right)$

LBE
70 bits

# Why we need loglog(n) bits?

- we have $L$ urns
- we have $2 \leq Q \leq n$ balls ($Q$ is fixed)
- $p_i$ ($i = 1, \ldots, L$) is the probability of choosing $i$-th urn
- $\Sigma_L = \{(p_1, \ldots, p_L) \in [0,1]^L : p_1 + \ldots + p_L = 1\}$
- $S_{\vec{p},Q}$ : at least one urn contains **precisely one** ball

$$\mathrm{MSP}\,(L,n) = \max_{\vec{p}\in\Sigma_L} \min_{2\leq Q\leq n} \Pr[S_{\vec{p},Q}]\ .$$

**Theorem**

*For arbitrary $L \geq 1$ and $n \geq 2$, we have*

$$\mathrm{MSP}\,(L,n) < \frac{L-1}{\mathrm{H_n} - 1}\ .$$

# Why we need loglog(n) bits?

Sketch of proof: part I

1. put: $S_{\vec{p},Q,i}$ = $i$-th urn contains exactly one ball"
2. then: $\Pr[S_{\vec{p},Q,i}] = Qp_i(1-p_i)^{Q-1}$
3. observe that: $\Pr[S_{\vec{p},Q}] \leq \sum_{i=1}^{L} \Pr[S_{\vec{p},Q,i}]$
4. D. E. Willard (1986) TRICK: define $f(\vec{p}) = \sum_{Q=2}^{n} \frac{\Pr[S_{\vec{p},Q}]}{Q}$
5. calculate

$$f(\vec{p}) \leq \sum_{Q=2}^{n} \sum_{i=1}^{L} \frac{\Pr[S_{\vec{p},Q,i}]}{Q} = \sum_{Q=2}^{n} \sum_{i=1}^{L} p_i(1-p_i)^{Q-1} \leq$$

$$\sum_{Q=2}^{n} \sum_{i=1}^{\infty} p_i(1-p_i)^{Q-1} = \ldots = L-1$$

# Why we need loglog(n) bits?

Sketch of proof : part II

1. take best $p^*$:

$$\min_{2\leq Q\leq n} \Pr[S_{p^*,Q}] = \max_{\vec{p}} \min_{2\leq Q\leq n} \Pr[S_{\vec{p},Q}] =: M$$

2. calculate

$$f(p^*) = \sum_{Q=2}^{n} \frac{\Pr[S_{p^*,Q}]}{Q} \geq \sum_{Q=2}^{n} \frac{M}{Q} = M(H_n - 1)$$

So we have

$$L - 1 \geq f(p^*) \geq \mathrm{MSP}\left(L, n\right)\left(\mathrm{H_n} - 1\right).$$

# Why we need loglog(n) bits?

Suppose now the our urns are indexed by $\{0,1\}^K$. They are used for distribution random bits.

**Theorem**

*If* $K < \log_2(\frac{1}{2}\ln n)$ *then* $\mathrm{MSP}\left(2^K, n\right) \leq \frac{1}{2}$.

# THANK YOU !!!