Analytic approach for reflected Brownian motion in the quadrant

AofA 2016

Sandro Franceschi
Joint work with Irina Kourkova and Kilian Raschel

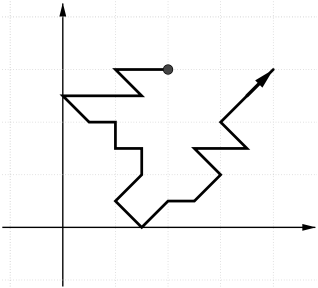
LPMA, University Pierre et Marie Curie & LMPT, University of Tours

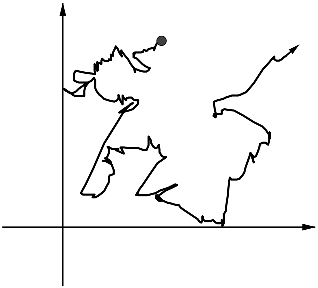
Kraków, Poland, July 4–8, 2016

AofA'16

Introduction

Random processes in the quarter plane:





- Discrete case (random walk) is studied a lot, remarkable exact formulas exist, it is popular in combinatorics.
- Continuous case (Brownian motion) serves as an approximation of large queuing networks.

Introduction

Goals:

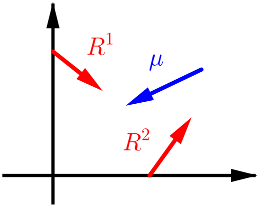
- Extend to the continuous case the analytic method developped by Malyshev in the seventies for the discrete case,
- **Compute explicitely** generating function of the **stationary distribution** *thanks to boundary value problems*,
- Study the **asymptotics of the stationary distribution** *thanks to saddle point methods*.

Plan

Reflected brownian motion in $\mathbb{R}_+^2$

Let

$$\begin{cases} (W_t)_{t\in\mathbb{R}^+} \text{ a planar brownian motion of covariance matrix } \Sigma \\ \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \in \mathbb{R}^2 \text{ a drift} \\ R = (R_1, R_2) = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2} \text{ a reflection matrix} \end{cases}$$



Reflected Brownian motion in $\mathbb{R}_+^2$

Definition

Let us define B_t the reflected Brownian motion in the quadrant as

$$B_t = B_0 + W_t + \mu t + R L_t \ \in \mathbb{R}_+^2$$

where L_t^i is a continuous non-decreasing process, that increases only when the process touches the boundary. (L_t **is a local time***)*

Reflected Brownian motion in $\mathbb{R}_+^2$

Definition

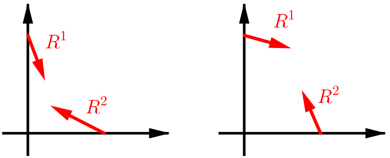
Let us define B_t the reflected Brownian motion in the quadrant as

$$B_t = B_0 + W_t + \mu t + R L_t \ \in \mathbb{R}_+^2$$

where L_t^i is a continuous non-decreasing process, that increases only when the process touches the boundary. (L_t **is a local time**)

Theorem (Reiman, Taylor, Williams, 1988 and 1993)

Such a process **exists** iff $r_{11} > 0$, $r_{22} > 0$ and either $r_{12}, r_{21} > 0$ or $r_{11} r_{22} - r_{12} r_{21} > 0$.



Process doesn't exist   Process exists

Recurrence criterion

Definition

B_t is said to be **recurrent positive** *if for all neighbourhood of zero $V \subset \mathbb{R}_+^2$ we have $\mathbb{E}[\tau_V] < \infty$ where $\tau_V = \inf\{t \geqslant 0 : B_t \in V\}$.*

Proposition (D. Hobson and L. Rogers, 1993)

The process and its **stationary distribution exist** *and is unique iff:*

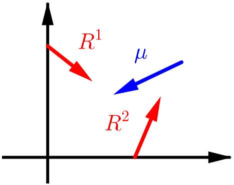
$$r_{11} > 0, \quad r_{22} > 0, \quad r_{11}r_{22} - r_{12}r_{21} > 0,$$

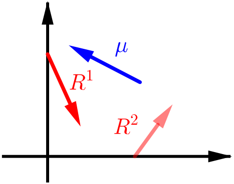
$$r_{22}\mu_1 - r_{12}\mu_2 < 0, \quad r_{11}\mu_2 - r_{21}\mu_1 < 0.$$

Recurrence criterion

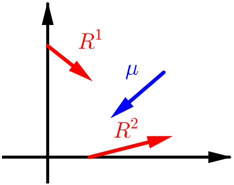
$$r_{11} > 0,\ r_{22} > 0,\ r_{11}r_{22} - r_{12}r_{21} > 0,\ r_{22}\mu_1 - r_{12}\mu_2 < 0,\ r_{11}\mu_2 - r_{21}\mu_1 < 0.$$

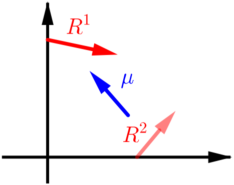
Recurrent cases

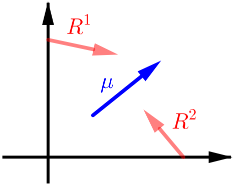




Transient cases







Stationary distribution and boundaries

Let π be the stationary distribution (or invariant measure) on $\mathbb{R}_+^2$ of the relecting Brownian motion B.

- Thanks to ergodic theorems, the invariant measure of the set $A \in \mathbb{R}_+^2$ is the average of the time proportion spent in A:

$$\pi(A) = \lim_{t\to\infty} \mathbb{E}[\frac{1}{t}\int_0^t \mathbb{1}_A(B_u)\mathrm{d}u]$$

- What about the boundaries?

Stationary distribution and boundaries

Let π be the stationary distribution (or invariant measure) on $\mathbb{R}_+^2$ of the relecting Brownian motion B.

- Thanks to ergodic theorems, the invariant measure of the set $A \in \mathbb{R}_+^2$ is the average of the time proportion spent in A:

$$\pi(A) = \lim_{t\to\infty} \mathbb{E}[\frac{1}{t}\int_0^t \mathbb{1}_A(B_u)\mathrm{d}u]$$

- What about the boundaries? We define ν_1 a measure on a boundary, such that for $A \in \{0\} \times \mathbb{R}$

$$\nu_1(A) = \mathbb{E}_\pi[\frac{1}{t}\int_0^t \mathbb{1}_A(B_u)\mathrm{d}L_u^1].$$

Similarly we define ν_2. We notice that ν_1 et ν_2 are a kind of *stationary distribution (or invariant measure) on the boundaries*.

Generating function of the stationary distribution

In the discrete case the generating functions of the stationary distribution $\pi_{i,j}$ on $\mathbb{Z}_+^2$ is the generating series $\sum_{\mathbb{Z}_+^2} \pi_{i,j} x^i y^j$.

Generating function of the stationary distribution

In the discrete case the generating functions of the stationary distribution $\pi_{i,j}$ on $\mathbb{Z}_+^2$ is the generating series $\sum_{\mathbb{Z}_+^2} \pi_{i,j} x^i y^j$.

- In the continuous case the generating function of the stationary distribution is the **Laplace transform**:

$$\phi(\theta) = \phi(\theta_1, \theta_2) = \iint_{\mathbb{R}_+^2} e^{\theta_1 x + \theta_2 y} \pi(x, y) \mathrm{d}x \mathrm{d}y$$

Generating function of the stationary distribution

In the discrete case the generating functions of the stationary distribution $\pi_{i,j}$ on $\mathbb{Z}_+^2$ is the generating series $\sum_{\mathbb{Z}_+^2} \pi_{i,j} x^i y^j$.

- In the continuous case the generating function of the stationary distribution is the **Laplace transform**:

$$\phi(\theta) = \phi(\theta_1, \theta_2) = \iint_{\mathbb{R}_+^2} e^{\theta_1 x + \theta_2 y} \pi(x,y) \mathrm{d}x \mathrm{d}y$$

- On the boundaries we define in an analogous way the following generating functions:

$$\phi_2(\theta_1) = \int_{\mathbb{R}_+} e^{\theta_1 x} \nu_2(x) \mathrm{d}x$$

$$\phi_1(\theta_2) = \int_{\mathbb{R}_+} e^{\theta_2 y} \nu_1(y) \mathrm{d}y$$

Plan

Functional equation

This equation binds the different generating functions.

Theorem

$$\gamma(\theta)\phi(\theta) = \gamma_1(\theta)\phi_1(\theta_2) + \gamma_2(\theta)\phi_2(\theta_1)$$

where

$$\begin{cases} \gamma(\theta) = -\frac{1}{2}(\sigma_{11}\theta_1^2 + \sigma_{22}\theta_2^2 + 2\sigma_{12}\theta_1\theta_2) - (\mu_1\theta_1 + \mu_2\theta_2), \\ \gamma_1(\theta) = \langle R^1|\theta\rangle = r_{11}\theta_1 + r_{21}\theta_2, \\ \gamma_2(\theta) = \langle R^2|\theta\rangle = r_{12}\theta_1 + r_{22}\theta_2. \end{cases}$$

It is an equation which connects what happens inside the quarter plane and on its boundaries.
The function γ is called the kernel.

Proof of the functional equation

Remark: The following relationships characterize the stationary distribution in different cases:

- $\pi(P - I) = 0$ for Markov chains,
- $\pi Q = 0$ for continuous time Markov chains,
- $\int \mathcal{G} f d\pi = 0$ for Markov processes where $\mathcal{G}$ is the generator.

Proof of the functional equation

Remark: The following relationships characterize the stationary distribution in different cases:

- $\pi(P - I) = 0$ for Markov chains,
- $\pi Q = 0$ for continuous time Markov chains,
- $\int \mathcal{G} f d\pi = 0$ for Markov processes where $\mathcal{G}$ is the generator.

The counterpart for the reflected Brownian motion in the quadrant is the "**basic adjoint relationship**":

$$\forall f \in \mathcal{C}_b^2(\mathbb{R}_+^2) \quad \int_{\mathbb{R}_+^2} \mathcal{G} f(z)\pi(\mathrm{d}z) + \sum_{i=1,2} \int_{\mathbb{R}_+^2} D_i f(z)\nu_i(\mathrm{d}z) = 0$$

where the generator "inside" the quadrant is

$$\mathcal{G}f(z) = \frac{1}{2}\sum_{i,j=1}^{2} \sigma_{i,j}\frac{\partial^2 f}{\partial z_1 \partial z_2}(z) + \sum_{i=1}^{2}\mu_i \frac{\partial f}{\partial z_i}(z)$$

and for $i = 1, 2$ the generators on the boundaries are

$$D_i f(x) = \langle R^i | \nabla f \rangle.$$

where the generator "inside" the quadrant is

$$\mathcal{G}f(z) = \frac{1}{2}\sum_{i,j=1}^{2} \sigma_{i,j} \frac{\partial^2 f}{\partial z_1 \partial z_2}(z) + \sum_{i=1}^{2} \mu_i \frac{\partial f}{\partial z_i}(z)$$

and for $i = 1, 2$ the generators on the boundaries are

$$D_i f(x) = \langle R^i | \nabla f \rangle.$$

$\hookrightarrow$ We just have to take $f = e^{\langle \theta | \cdot \rangle}$ in the basic adjoint relationship to obtain the functional equation. Indeed

$$\int_{\mathbb{R}_+^2} \mathcal{G} e^{\langle \theta | z \rangle} \pi(\mathrm{d}z) + \sum_{i=1,2} \int_{\mathbb{R}_+^2} D_i e^{\langle \theta | z \rangle} \nu_i(\mathrm{d}z) = 0$$

gives

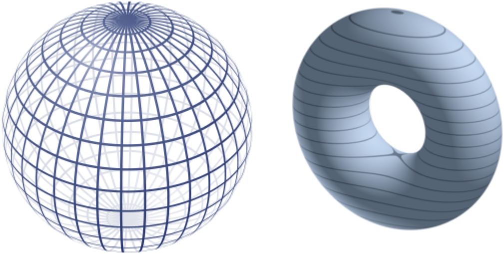
$$\gamma(\theta)\phi(\theta) = \gamma_1(\theta)\phi_1(\theta_2) + \gamma_2(\theta)\phi_2(\theta_1).$$

Riemann surface

- We introduce the Riemann surface $\mathcal{S}$ as the zeros of the kernel γ :

$$\boxed{\mathcal{S} = \{(\theta_1, \theta_2) \in \mathbb{C}^2 : \gamma(\theta_1, \theta_2) = 0)\}}$$

- On the Riemann surface $\mathcal{S}$ the first part of the functional equation disappear and we have $0 = \gamma_1 \phi_1 + \gamma_2 \phi_2$.
- Here $\mathcal{S}$ is a sphere. *In the discrete case it's a torus*.



Key steps of the analytic method

- Find a **functional equation**
- Study the kernel and the Riemann surface (sphere, torus, ...)
- Continue meromorphically the generating functions on the Riemann surface
- Establish a **boundary value problem** *to find explicit expressions*
- Study the singularities and use the **saddle point method** *to determine the asymptotics*
- Introduce the group of the process, its finitness is linked to the *algebraic nature of the solution*

Plan

What is a BVP with shift?

A boundary value problem with shift is made of two conditions:

- a regularity condition on some set
- a boundary condition with shift

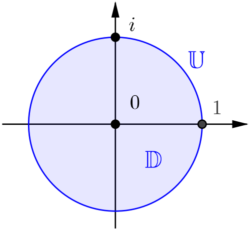
What is a BVP with shift?

A boundary value problem with shift is made of two conditions:

- a regularity condition on some set
- a boundary condition with shift

Example:

1. f is meromorphic on the unit disc $\mathbb{D}$ and has only one pole of order one in 0
2. $f(\bar{z}) = f(z)$ for $z \in \mathbb{U}$ the unit circle



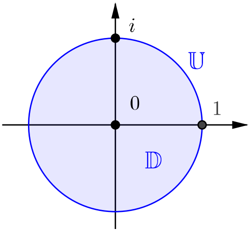
What is a BVP with shift?

A boundary value problem with shift is made of two conditions:

- a regularity condition on some set
- a boundary condition with shift

Example:

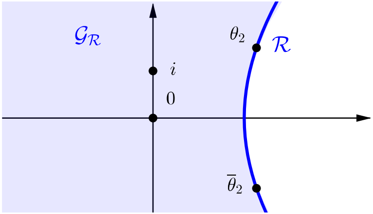
1. f is meromorphic on the unit disc $\mathbb{D}$ and has only one pole of order one in 0
2. $f(\bar{z}) = f(z)$ for $z \in \mathbb{U}$ the unit circle



The solution of this BVP with shift is $f(z) = z + \frac{1}{z}$. It is a gluing function which glues together the upper and the lower part of $\mathbb{U}$.

Boundary of the BVP

$$\mathcal{R} = \{\theta_2 \in \mathbb{C} : \gamma(\theta_1, \theta_2) = 0 \text{ where } \theta_1 \in \mathbb{R}_- \text{ and } \theta_2 \notin \mathbb{R}\}$$



- The curve $\mathcal{R}$ is an hyperbola symetric with respect to the x-axis (and $\mathcal{G}_\mathcal{R}$ is the blue domain).
- If $\theta_2 \in \mathcal{R}$ and $\gamma(\theta_1, \theta_2) = 0$ then $\gamma(\theta_1, \bar{\theta}_2) = 0$.

Statement of the BVP

Lemma

The function ϕ_1 satisfy the following BVP with shift:

1. ϕ_1 *is meromorphic on* $\mathcal{G}_{\mathcal{R}}$ *with at most one pole* p *of order* 1*, and is bounded at infinity;*
2. ϕ_1 *is continuous on* $\overline{\mathcal{G}_{\mathcal{R}}} \setminus \{p\}$ *and*

$$\boxed{\phi_1(\overline{\theta_2}) = G(\theta_2)\phi_1(\theta_2), \qquad \forall \theta_2 \in \mathcal{R}.}$$

Statement of the BVP

Lemma

The function ϕ_1 satisfy the following BVP with shift:

1. ϕ_1 *is meromorphic on* $\mathcal{G}_\mathcal{R}$ *with at most one pole* p *of order* 1, *and is bounded at infinity;*
2. ϕ_1 *is continuous on* $\overline{\mathcal{G}_\mathcal{R}} \setminus \{p\}$ *and*

$$\boxed{\phi_1(\overline{\theta_2}) = G(\theta_2)\phi_1(\theta_2), \qquad \forall \theta_2 \in \mathcal{R}.}$$

where we have defined for $\theta_2 \in \mathcal{R}$

$$G(\theta_2) = \frac{\gamma_1}{\gamma_2}(\Theta_1^-(\theta_2), \theta_2)\frac{\gamma_2}{\gamma_1}(\Theta_1^-(\theta_2), \overline{\theta_2}).$$

with the bi-valued algebraic function $\Theta_1^-(\theta_2)$ associated to $\mathcal{S}$ defined by $\gamma(\Theta_1^-(\theta_2), \theta_2) = 0$.

Proof of the BVP

Let $\underline{\theta_2 \in \mathcal{R}}$ and θ_1 such that $(\theta_1, \theta_2) \in \mathcal{S}$ and $(\theta_1, \overline{\theta_2}) \in \mathcal{S}$.
Thus

$$0 = \gamma_1(\theta_1, \theta_2)\phi_1(\theta_2) + \gamma_2(\theta_1, \theta_2)\phi_2(\theta_1)$$

$$0 = \gamma_1(\theta_1, \overline{\theta_2})\phi_1(\overline{\theta_2}) + \gamma_2(\theta_1, \overline{\theta_2})\phi_2(\theta_1)$$

We deduce that for $\theta_2 \in \mathcal{R}$

$$\Rightarrow \quad \phi_1(\overline{\theta_2}) = \underbrace{\frac{\gamma_1}{\gamma_2}(\theta_1, \theta_2)\frac{\gamma_2}{\gamma_1}(\theta_1, \overline{\theta_2})}_{G(\theta_2)}\phi_1(\theta_2)$$

Gluing function

The conformal gluing function w glues together the upper and lower parts of the hyperbola $\mathcal{R}$. It can be expressed in terms of the *generalized Chebyshev polynomial*,

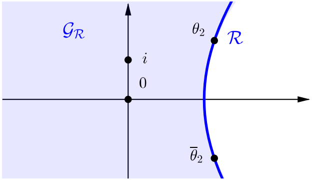
$$T_a(x) = \cos(a \arccos(x)) = \frac{1}{2}\left\{\left(x+\sqrt{x^2-1}\right)^a + \left(x-\sqrt{x^2-1}\right)^a\right\}$$

as follow:

$$w(\theta_2) = T_{\frac{\pi}{\beta}}\left(-\frac{2\theta_2 - (\theta_2^+ + \theta_2^-)}{\theta_2^+ - \theta_2^-}\right),$$

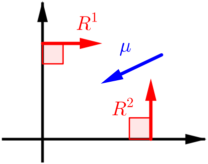
where we put

$$\beta = \arccos - \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}.$$



Resolution of the BVP: case of orthogonal reflexion

In the case of an orthogonal reflexion we can reduce the problem to a more simple BVP with $G = 1$. Thanks to an invariant lemma and to the gluing function we obtain:



Theorem (F. , Raschel, 2016)

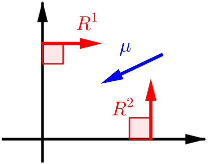
Let R be the identity matrix. The Laplace transform ϕ_1 is worth

$$\phi_1(\theta_2) = \frac{-\mu_1 w'(0)}{w(\theta_2) - w(0)}\theta_2,$$

where w is the gluing function.

Resolution of the BVP: case of orthogonal reflexion

In the case of an orthogonal reflexion we can reduce the problem to a more simple BVP with $G = 1$. Thanks to an invariant lemma and to the gluing function we obtain:



Theorem (F. , Raschel, 2016)

Let R be the identity matrix. The Laplace transform ϕ_1 is worth

$$\phi_1(\theta_2) = \frac{-\mu_1 w'(0)}{w(\theta_2) - w(0)} \theta_2,$$

where w is the gluing function.

- In the **general case** we are able to find explicitly ϕ_1 in term of Cauchy integral and the gluing function.

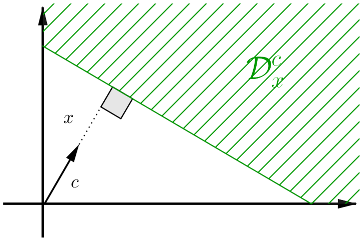
Asymptotics of the stationary distribution

Theorem (Dai-Miyazawa 2011)

If $\mathcal{D}_x^c$ is the hatched set

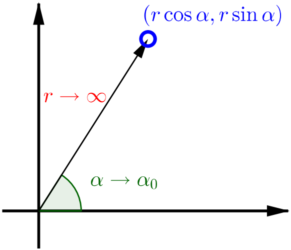
$$\pi(\mathcal{D}_x^c) \sim_{x\to\infty} b x^{\kappa_c} e^{-\alpha_c x}$$

where constants α_c and κ_c can be explicitly computed and κ_c takes one of the values : $-3/2$, $-1/2$, 0 ou 1.



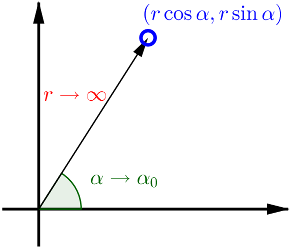
Method to determine the asymptotics

We try to make an asymptotic developpement in all directions, that is of $\pi(r\cos\alpha, r\sin\alpha)$ when $r\to\infty$ and $\alpha\to\alpha_0$.



Method to determine the asymptotics

We try to make an asymptotic developpement in all directions, that is of $\pi(r\cos\alpha, r\sin\alpha)$ when $r\to\infty$ and $\alpha\to\alpha_0$.



Method:

- Continue meromorphically the generating functions on $\mathcal{S}$
- Study the singularities on $\mathcal{S}$
- Inverse the Laplace transforms
- Use saddle point methods on the Riemann surface $\mathcal{S}$

Asymptotics in all directions

Theorem (F. , Kourkova, 2016)

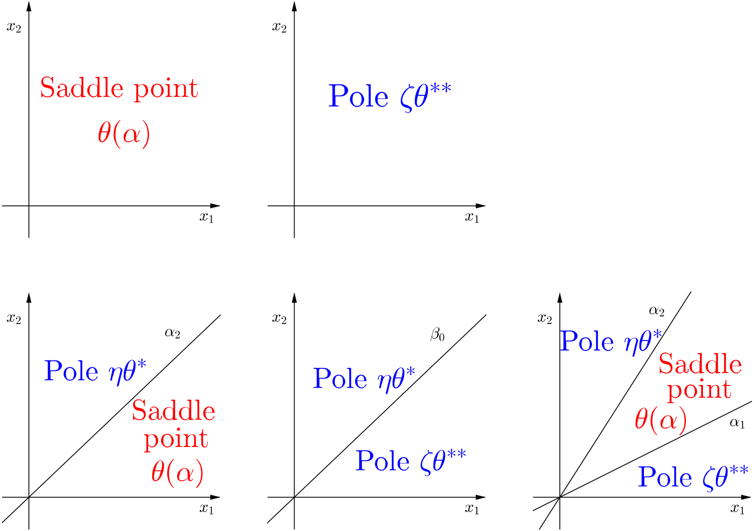
Let $\alpha_0 \in (0, \pi/2)$. When $r \to \infty$ and $\alpha \to \alpha_0$ according to the parameters we have

$$\pi(r\cos\alpha, r\sin\alpha) = (1 + o(1)) \cdot \begin{cases} \frac{C_0}{\sqrt{r}} e^{-r\langle(\cos\alpha,\sin\alpha)|\theta(\alpha)\rangle}, \\ C_1 e^{-r\langle(\cos\alpha,\sin\alpha)|\eta\theta^*\rangle}, \\ C_2 e^{-r\langle(\cos\alpha,\sin\alpha)|\zeta\theta^{**}\rangle}, \end{cases}$$

where C_0, C_1 and C_2 are constants which can be computed in function of ϕ_1, ϕ_2 and the parameters.

The decay rates $\eta\theta^*$ and $\zeta\theta^{**}$ come from the poles and $\theta(\alpha)$ from the saddle point.
We are able to find the **full asymptotic development**.

Asymptotics according to the direction



Thank you for your attention!

J. G. DAI AND M. MIYAZAWA - "Reflecting brownian motion in two dimensions: Exacts asymptotics for the stationary distribution", *Stochastic Systems,* **1** (2011), p. 146-208.

G. FAYOLLE, R. IASNOGORODSKI AND V. MALYSHEV - *Random walks in the quarter-plane,* Application of Mathématics (New York), vol. 40, Springer, (1999).

S. FRANCESCHI AND K. RASCHEL - "Tutte-s invariant approach for Brownian motion reflected in the quadrant", *arXiv: 1602.03054* (2016)

S. FRANCESCHI AND I. KURKOVA - "Asymptotic expansion of stationary distribution for reflected brownian motion in the quadrant via analytic approach" *arXiv:1604.02918* (2016)

I. KURKOVA AND V. MALYSHEV - "Martin boundary and elliptic curves", *Markov Process and Related Fields,* **4** (1998), p. 203-272.

R. J. WILLIAMS - "Semimartingale reflecting Brownian motions in the orthant.", *Stochastic Networks* , **13** (1995).

Kernel

- The kernel γ can be written as $\gamma(\theta_1, \theta_2) = a(\theta_1)\theta_2^2 + b(\theta_1)\theta_2 + c(\theta_1)$. The two branches are given by
$$\Theta_2^{\pm}(\theta_1) = \frac{-b(\theta_1) \pm \sqrt{d(\theta_1)}}{2a(\theta_1)},$$
where $d(\theta_1) = b^2(\theta_1) - 4a(\theta_1)c(\theta_1)$ is the discriminant.
- The polynomial d has two roots, called $\theta_1^{\pm}$ which are the branching points of Θ_2.
- We notice that d is negative on $(-\infty, \theta_1^-) \cup (\theta_1^+, \infty)$. Branches $\Theta_2^{\pm}$ take complex conjugate values on this set.

Group of the process

- It is the group $\langle\zeta,\eta\rangle$ generated by ζ and η, given by

$$\zeta(\theta_1,\theta_2)=\left(\theta_1,\frac{c(\theta_1)}{a(\theta_1)}\frac{1}{\theta_2}\right),\quad \eta(\theta_1,\theta_2)=\left(\frac{\widetilde{c}(\theta_2)}{\widetilde{a}(\theta_2)}\frac{1}{\theta_1},\theta_2\right).$$

- By construction if $\gamma(\theta_1,\theta_2)=0$ then $\gamma(\zeta(\theta_1,\theta_2))=\gamma(\eta(\theta_1,\theta_2))=0$. They are automorphism of $\mathcal{S}$. We have

$$\zeta(\theta_1,\Theta_2^+(\theta_1))=\zeta(\theta_1,\Theta_2^-(\theta_1))$$

- The algebicity of generating functions depends of the finiteness of the group.