PERIODIC OSCILLATIONS OF DIVIDE-AND-CONQUER RECURRENCES WITH BALANCED PART SIZES

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SIMPLEST DIVIDE-AND-CONQUER RECURRENCE

$$f(n) = f\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) + f\left(\left\lceil \tfrac{n}{2} \right\rceil\right) + g(n)$$

Q: iff-condition on $g(n)$ for $\begin{cases} f(n) = O(n) \\ f(n) \sim nP(\log_2 n) \end{cases}$ ***?***

Concrete examples: **mergesort, matrix multiplication, sorting n/ws, fast Fourier transform, random trees, sub-additivity, digital sums, convex-hull algorithms, combinatorial sequences, ...**

Hammersley & Grimmett (1974):

$f(n) = \min_k\{f(k) + f(n-k)\} + g(n)$

- $g \downarrow \Longrightarrow k = 1$
- $g \uparrow$ **& convex** $\Longrightarrow k = \left\lfloor \frac{n}{2} \right\rfloor$
- $g \uparrow$ **& concave** $\Longrightarrow k = 2^{\left\lfloor \log_2 \frac{2}{3} n \right\rfloor}$

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SOLVING THE RECURRENCE

$f(n) = f(\lfloor \frac{n}{2} \rfloor) + f(\lceil \frac{n}{2} \rceil) + g(n)$ **written as** $\Lambda[f](n) = g(n)$

Master theorem

$$f(n) = \begin{cases} O(n), & \textbf{if } g(n) = O(n^{1-\varepsilon}) \\ O(n \log n), & \textbf{if } g(n) \asymp n \\ O(g(n)), & \textbf{if } g(n) = \Omega(n^{1+\varepsilon}) \ \& \ \textbf{regular varying} \end{cases}$$

Master theorem: a brief history

- **Aho, Hopcroft & Ullmann (1974, 1983):** $g(n) = O(1)$ & $O(n)$
- **Bentley, Haken & Saxe (1980): the 1st paper**
- **BHS's framework presented in many books since: Mehlhorn (1984), Purdom & Brown (1985), ...**
- **Cormen, Leiserson & Rivest (1990): use "Master theorem"**
- **Most extensions along two lines:**
  - **more general recurrences**
  - **more general toll functions**

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MASTER THEOREM

General recurrences

Bentley, Haken & Saxe (1980) **Verma (1994), Mogos (2013)**	$f(n) = cf\left(\frac{n}{r}\right) + g(n)$
Wang & Fu (1993, 1996)	$f(n) = c_n f(r_n) + g(n)$
Akra & Bazzi (1998) **Leighton (1996)** **Kao (1997), Verma (1997)** **Schöning (2000), Yap (2011)**	$f(x) = \sum_{1 \leqslant k \leqslant r} c_k f(a_k x) + g(x)$
Roura (2001)	$f(n) = \sum_{1 \leqslant k \leqslant r} c_{n,k} f(\rho_{n,k}) + g(n)$
Drmota & Szpankowski (2013)	$f(n) = \sum_{1 \leqslant k \leqslant r} a_k f(\lfloor p_k n + \delta_k \rfloor)$ $+ \sum_{1 \leqslant k \leqslant r} b_k f(\lceil p_k n + \delta'_k \rceil) + g(n)$

Some number theoretic recurrences

Kalmár (1931)	$f(n) = 1 + \sum_{k \geqslant 2} f(\lfloor n/k \rfloor)$
Hille (1936) **Erdős (1941)** . . . **H. & Janson (2010)**	$f(n) = 1 + \sum_{k \geqslant 1} f(\lfloor n/a_k \rfloor)$
Erdős et al. (1987)	$f(n) = 1 + \sum_{1 \leqslant k \leqslant r} c_k f(\lfloor n/a_k \rfloor)$

ASYMPTOTIC LINEARITY

$$f(n) = f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + f\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n)$$

Q: what $g(n)$ implies $f(n) \asymp n$?

Aho et al. (1975)	$g(n) = O(1)$
Bentley et al. (1980)	$g(n) = O\left(n^{1-\varepsilon}\right)$
Verma (1994) **Yap (2011)**	$g(n) = O\left(n(\log n)^{-1-\varepsilon}\right)$

Kieffer (2012): $g(n) = O(1) \implies f(n) = nP(\log_2 n) + o(n)$

Q: opt. condition? cv of $\sum_{k \geqslant 1} \frac{g(k)}{k^2}$ ***sufficient?***

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KNOWN CASES

Quicksort recurrence: $f(n) = \frac{2}{n}\sum_{k<n} f(k) + g(n)$

$$f(n) = 2(n+1)\sum_{1\leqslant k<n}\frac{g(k)}{(k+1)(k+2)} + g(n)$$

$$\boxed{f(n) = O(n)} \quad \textbf{iff} \quad \boxed{g(n) = O(n) \;\&\; \sum_{1\leqslant k\leqslant m}\frac{g(k)}{k^2} = O(1)} \qquad (\star)$$

Exactly the same iff-result $(\star)$ holds for $f(n) = \sum_{1\leqslant k<n}\pi_{n,k}f(k) + g(n)$

- **m-ary search trees ($m \geqslant 2$):** $\pi_{n,k} = m\frac{\binom{n-1-k}{m-2}}{\binom{n}{m-1}}$
- **median-of-$(2t+1)$ quicksort ($t \geqslant 0$):** $\pi_{n,k} = 2\frac{\binom{n-1-k}{t}\binom{k}{t}}{\binom{n}{2k+1}}$
- **quadtrees ($d \geqslant 1$):** $\pi_{n,k} = \sum_{0\leqslant j<n-k}\binom{n-1}{k}\binom{n-1-k}{j}\frac{(-1)^j}{(k+j+1)^d}$

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$(\star)$ FAILS FOR $\Lambda[f] = g$

$$g(n) = \begin{cases} \frac{2^\ell}{\ell}, & \textbf{if } n = 2^\ell \\ 0, & \textbf{otherwise} \end{cases} \implies \sum_{2\leqslant k\leqslant m} \frac{g(k)}{k^2} = O(1)$$

$$\textbf{but } f(2^m) = 2^m \sum_{1\leqslant k<m} \frac{1}{k} = O(2^m \log m)$$

Boundedness of $\sum_{1\leqslant k\leqslant m} \frac{g(k)}{k^2}$ also appeared in

- **DaC algos in computational geometry: Devroye (1983, 1994), Clarkson & Shor (1989)**
- **Linearity of subadditive functions: Hammersley (1962), Hammersley & Grimmett (1974)**

WHAT ABOUT THE BDDNESS OF $\sum_{0\leqslant j\leqslant m} \frac{g(2^j)}{2^j}$?

Another known case (heap & queue-mergesort):

$$f(n) = f\left(2^{\lfloor \log_2 \frac{2}{3}n \rfloor}\right) + f\left(n - 2^{\lfloor \log_2 \frac{2}{3}n \rfloor}\right) + g(n)$$

$$\boxed{f(n) = O(n)} \quad \textbf{iff} \quad \boxed{g(n) = O(n) \quad \& \quad \sum_{1\leqslant j\leqslant m} \frac{g(2^j)}{2^j} = O(1)}$$

Bddness of $\sum_{0\leqslant j\leqslant m} \frac{g(2^j)}{2^j}$ sufficient for asymp. linearity of $\Lambda[f] = g$?

A counterexample

$$g(n) = \begin{cases} \frac{2^k}{k}, & \text{if } n = 3\cdot 2^k, k \geqslant 1 \\ 0, & \text{otherwise} \end{cases} \implies \begin{cases} \sum_{0\leqslant j\leqslant m} \frac{g(2^j)}{2^j} = O(1) \\ f(n) = O(n \log\log n) \end{cases}$$

Cauchy's condensation test

If $\frac{g(n)}{n^2} \uparrow$, then $\boxed{\sum_{k\geqslant 1} \frac{g(k)}{k^2} \text{ converges}}$ iff $\boxed{\sum_{j\geqslant 0} \frac{g(2^j)}{2^j} \text{ converges}}$.

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OUR APPROACH: THE KEY IDEA

$$\begin{cases} f(2n) = 2f(n) + g(2n) \\ f(2n+1) = f(n) + f(n+1) + g(2n+1) \end{cases}$$

The key: *linear interpolation*

Extend $f(n)$ **(**$n \in \mathbb{Z}^+$**) to** $f(x)$ **(real** $x \geqslant 1$**) by:**

$$f(x) := f(\lfloor x \rfloor) + \{x\}(f(\lfloor x \rfloor + 1) - f(\lfloor x \rfloor))$$

Then $\boxed{f(x) = 2f\left(\frac{x}{2}\right) + g(x)}$ **for** $x \geqslant 2$**.**

For $x \geqslant 1$ & $0 \leqslant m \leqslant L_x := \lfloor \log_2 x \rfloor$

$$f(x) = \sum_{0 \leqslant k < m} 2^k g(2^{-k}x) + 2^m f(2^{-m}x)$$

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The first iff condition: $(i) \equiv (ii)$

(*i*) $f(n) = nP(\log_2 n) + o(n)$ **as** $n \to \infty$**, for some continuous & 1-periodic function** P **on** $\in \mathbb{R}$**.**

(*ii*) $G_k(t) := \sum_{1\leqslant j\leqslant k} 2^{-j} g(2^j t)$ ***converges uniformly*** **for** $t \in [1,2]$ **as** $k \to \infty$**.**

When both hold, the 1-periodic function P **is given by**

$$P(t) := f(1) + \sum_{k\in\mathbb{Z}} \frac{g(2^{k+\{t\}})}{2^{k+\{t\}}} \qquad \textbf{(exp. cv)}$$

$$= f(1) + \frac{D'(0)}{\log 2} + \frac{1}{\log 2}\sum_{k\neq 0} \frac{D(\chi_k)}{\chi_k(\chi_k+1)} e^{2k\pi i t} \qquad \textbf{(poly cv)}$$

Furthermore, $\boxed{f(n) \equiv nP(\log_2 n) - G(n)}$ **(an identity!).**

$$D(s) := \sum_{j\geqslant 2} g(j)\left((j+1)^{-s} - 2j^{-s} + (j-1)^{-s}\right)$$

NEW RESULTS

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O-BOUND & A SIMPLER SUFFICIENT CONDITION

iff-condition for O-bound

$$\boxed{f(n) = O(n)} \quad \textbf{iff} \quad \boxed{\sum_{1 \leqslant k \leqslant m} 2^{-k} g(2^k t) = O(1)} \; (t \in [1,2])$$

An easier-to-apply sufficient condition

$$f(n) = f\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) + f\left(\left\lceil \tfrac{n}{2} \right\rceil\right) + g(n)$$

$$g(n) = O\left(\frac{n}{(\log n)^{1+\varepsilon}}\right), \quad (\varepsilon > 0)$$

$$\Longrightarrow f(n) = nP(\log_2 n) - \sum_{k \geqslant 1} 2^{-k} g(2^k n)$$

here
- P **has an abs. conv. Fourier series**
- P **is of bounded variation**
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here $\begin{cases} P \text{ has an abs. conv. Fourier series} \\ P \text{ is of bounded variation} \\ P \text{ is Lipschitz continuous} \end{cases}$

$\Lambda[f] = g$: DIVERSE APPROACHES INTRODUCED

Other approaches

- ***Real analytic*** **(calculus, linear algebra, repertoire, functional iteration, etc.): Bentley et al. (1980), Aho et al. (1983), Purdom & Brown (1985), Greene & Knuth (1990), Cormen et al. (1990), Yap (2011), Dumas (2014), . . .**
- ***Renewal theory*****: Erdős et al. (1987)**
- ***Tauberian theorems*****: Flajolet et al. (1994), Drmota & Szpankowski (2013)**
- ***Complex analytic*****: finite difference, Mellin-Perron integral, residue calculus: Flajolet & Golin (1993, 1994), H. (1998), Grabner & H. (2005), . . .**

$$f(n) = f(1) + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{n^{s+1} D(s)}{s(s+1)(1-2^{-s})}\, \mathrm{d}s$$

- ***Fractal geometry & iterated function systems*****: Dube (1995, 2009), Oh (2012), Kieffer (2012), . . .**

100 Examples of

$$f(n) = f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + f\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n)$$

BOUNDED $g(n)$

Sloane's OEIS, Online Encyclopedia of Integer Sequences ($> 274,000$ items)

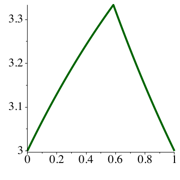
- **A005942, # cmps used to find min & max by DaC**
  $g(n) = 0$ **&** $f(0) = 2, f(1) = 4, f(2) = 6$

$$f(n) = \begin{cases} 4n - 2^{L_n}, & \textbf{if } 2^{L_n} \leqslant n < \frac{3}{2}2^{L_n} \\ 2n + 2^{L_n+1}, & \textbf{if } \frac{3}{2}2^{L_n} \leqslant n < 2^{L_n+1} \end{cases}$$

$$= nP(\log_2 n), \qquad P(t) = \begin{cases} 4 - 2^{-t} & t \in [0, \log_2 3 - 1] \\ 2 + 2^{1-t} & t \in [\log_2 3 - 1, 1]. \end{cases}$$

Other bounded cases

A005942, A006165, A007378, A060973, A079905, A079945, A080637, A080639, A080640, A080641, A080644, A080645, A080646, A080653, A080776, ...



A005942 = subword complexity of Thue-Morse seq.

BOUNDED $g(n)$

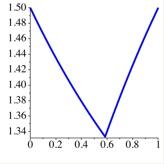
- **A080653,** $f(n)$ **equals the smallest nonnegative number** $> f(n-1)$ **such that the condition "**n **is in sequence only if** $f(n)$ **is even" is satisfied. It satisfies** $\Lambda[f] = g$ **with** $f(1) = 1$ **&**

$$\boxed{g(n) = 1 + \lfloor \log_2(n+1) \rfloor - \left\lfloor \log_2 \tfrac{4}{3}(n+1) \right\rfloor} \quad (n \geqslant 2)$$

(2^k **1's followed by** 2^k **0's for** $k \geqslant 0$**)**

$\bar{f}(n) := f(n-1) + 2 =$ **A007378** $\Longrightarrow \Lambda[\bar{f}] = 0$ **with** $\bar{f}(1) = 1$ **&** $\bar{f}(2) = 3$**. Thus** $\bar{f}(n) = nP(\log_2 n)$**,**

$$P(t) = \begin{cases} 1 + 2^{-1-t} & t \in [0, \log_2 3 - 1] \\ 2 - 2^{-t} & t \in [\log_2 3 - 1, 1] \end{cases}$$



BOUNDED $g(n)$

- **Mergesort variance (Flajolet & Golin, 1994):**
  $\Lambda[f] = g$

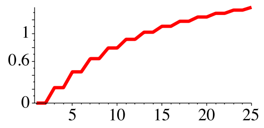
$$g(n) = \frac{2\left\lceil\frac{n}{2}\right\rceil^2\left(\left\lceil\frac{n}{2}\right\rceil - 1\right)}{\left(\left\lceil\frac{n}{2}\right\rceil + 1\right)^2\left(\left\lceil\frac{n}{2}\right\rceil + 2\right)} \quad (n \geqslant 2)$$

$f(n) = nP(\log_2 n) - G(n)$, $G(n) := \sum_{k\geqslant 1} 2^{-k} g(2^k n)$

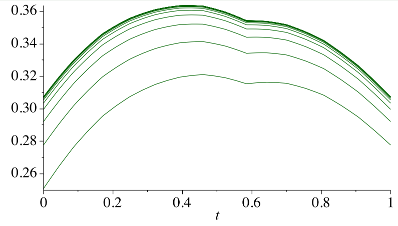
$$f(n) = nP(\log_2 n) - 2 - \sum_{k\geqslant 0} \frac{1}{2^k}\left(\frac{7}{2^k n + 1} - \frac{12}{2^k n + 2} - \frac{2}{(2^k n + 1)^2}\right)$$

where $P(t) := \sum_{k\in\mathbb{Z}} 2^{-k-t} g(2^{k+t}) = \sum_{k\geqslant 1} 2^{-k-t} g(2^{k+t})$, $t \in [0,1]$

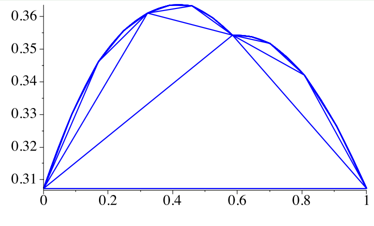
$g(x)$**:** $x = 0, \ldots, 200$



$x = 0, \ldots, 25$



$P(t)$ **by** $\sum_{k\geqslant 1} \frac{g(2^{k+t})}{2^{k+t}}$



$P(\{\log_2 n\})$ **by** $\frac{f(n)+G(n)}{n}$

Fourier expansion

$$\widehat{P}(0) = \frac{1}{\log 2}\int_2^\infty \frac{g(t)}{t^2}\,\mathrm{d}t = \frac{1}{\log 2}\sum_{m\geqslant 1}\frac{2m(5m^2+10m+1)}{(m+1)^2(m+2)^2(m+3)}\log\frac{2m+1}{2m}$$

$$\approx 0.34549\,32539\,59979\,17006\,74766\ldots$$

$$\widehat{P}(k) = \frac{2}{(\log 2)\chi_k(1+\chi_k)}\sum_{m\geqslant 1}\frac{m(2m^2+10m+1)}{(m+1)^2(m+2)^2(m+3)}\left(m^{-\chi_k} - (m+\tfrac{1}{2})^{-\chi_k}\right)$$

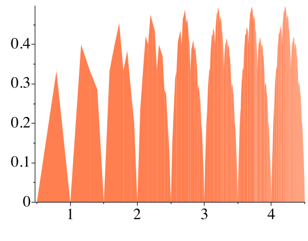
All known results (Flajolet & Golin, 1994, H. 1998) rederived

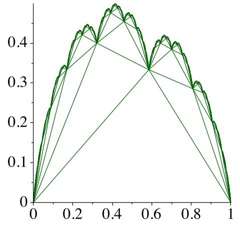
All cumulants doable by the same approach

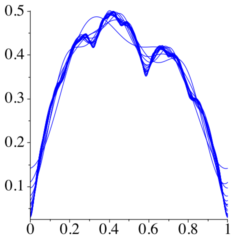
- **Oh & Kieffer (2010) (Lossless compression of balanced trees):** $g(n) = \mathbf{1}_{n \text{ is odd}}$ **for** $n \geqslant 3$ **&** $f(1) = 0$**. Thus** $f(n) = nP(\log_2 n)$**, where**

$$P(t) = \sum_{k \geqslant 1} \frac{g(2^{k+\{t\}})}{2^{k+\{t\}}}, \quad g(x) = \begin{cases} \{x\}, & \lfloor x \rfloor \text{ even} \\ 1 - \{x\}, & \lfloor x \rfloor \text{ odd} \end{cases}$$

$$P(t) = 2 - \log_2 \pi + \frac{1}{\log 2} \sum_{k \neq 0} \frac{1 + 2\zeta(\chi_k)}{\chi_k(\chi_k + 1)} e^{2k\pi i t}$$



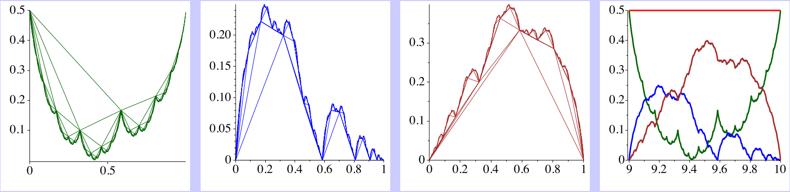




BOUNDED $g(n)$: A SENSIBILITY TEST

- **Consider** $\Lambda[f_j] = g_j$ **with** $g_j(n) := 1$ **if** $n \bmod 4 \equiv j$ **and** $f_j(0) = f_j(1) = 0$. **Then**

$$\begin{cases} f_0(n) = nP_0(\log_2 n) + 1 - \left\{\frac{n}{2}\right\} \\ f_1(n) = nP_1(\log_2 n) \\ f_2(n) = \left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2} - \left\{\frac{n}{2}\right\} \\ f_3(n) = nP_3(\log_2 n) \end{cases}$$



$\frac{f_0(n)+1-\{\frac{n}{2}\}}{n}$ $\quad \frac{f_1(n)}{n}$ $\quad \frac{f_3(n)}{n}$ $\quad P_j\ (j = 0, 1, 2, 3)$

Minor changes in g result in very different fluctuations

SUBLINEAR $g(n)$

Most examples from computational geometry

- **A class of balanced binary trees (Cha, 2012):** $g(n) = \lceil \log_2 n \rceil$ **(**$f(n) =$ **A213508).**

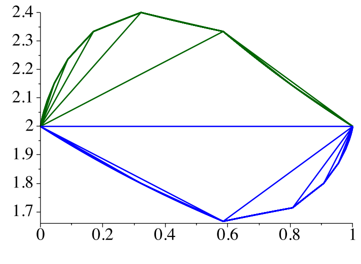
$$G(n) = \sum_{k \geqslant 1} 2^{-k} g(2^k n) = \lceil \log_2 n \rceil + 2$$
$$\Longrightarrow f(n) = nP(\log_2 n) - \lceil \log_2 n \rceil - 2$$

$$P(t) = 2^{1-\{t\}} + (1 - 2^{-\{t\}})\left(1 + 2^{-\{\log_2(2^{\{t\}}-1)\}} - \lfloor \log_2(2^{\{t\}} - 1) \rfloor\right)$$

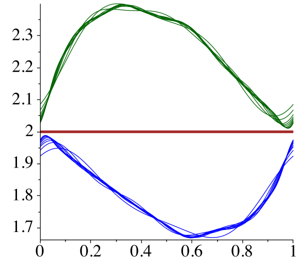
- $g(n) = \lfloor \log_2 n \rfloor \Longrightarrow f(n) = nP(\log_2 n) - \lfloor \log_2 n \rfloor - 2$

$$P(t) = 1 + 2^{1-\{t\}} - (2^{1-\{t\}} - 1)\left(2^{-\{\log_2(2-2^{\{t\}})\}} - \lfloor \log_2(2 - 2^{\{t\}}) \rfloor\right)$$

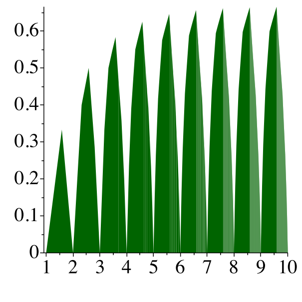
Few cases beyond bounded g for which P can be made explicit



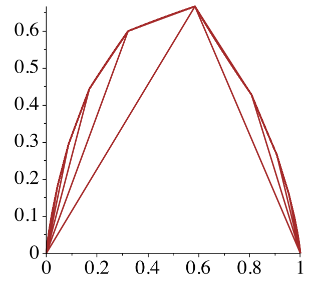
$P(t)$ **from** $\lceil \log_2 n \rceil$ **&** $\lfloor \log_2 n \rfloor$



$P(t)$ **by Fourier**



$\Lambda[f] = \lceil \log_2 n \rceil - \lfloor \log_2 n \rfloor$



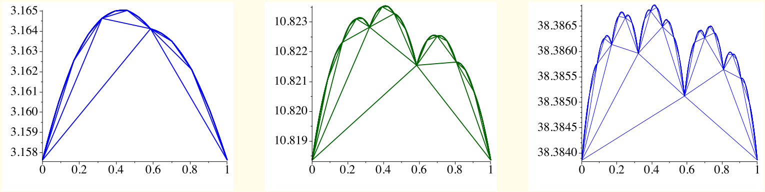
in $\{\log_2 n\}$

SUBLINEAR $g(n)$

- **Expected cost of a naive maxima-finding algorithm (Flajolet & Golin, 1993):** $g(n) = M_{\lfloor \frac{n}{2} \rfloor, d} M_{\lceil \frac{n}{2} \rceil, d} \asymp (\log n)^{2d-2}$ **($d \geqslant 2$)**

$$M_{n,d} = \sum_{1 \leqslant k \leqslant n} \binom{n}{k} \frac{(-1)^{k-1}}{k^{d-1}}$$

$$\Longrightarrow f(n) = nP(\log_2 n) - \underbrace{\sum_{k \geqslant 0} 2^{-k-1} M^2_{2^k n, d}}_{\asymp (\log n)^{2d-2}}$$



$d = 2$ $\quad$ $d = 3$ $\quad$ $d = 4$

$$g(n) = n^\tau, \tau \in (0,1) \Longrightarrow f(n) = nP(\log_2 n) - \frac{n^\tau}{2^{1-\tau} - 1}$$

LINEAR $g(n)$

Most cases: $g(n) = cn + \bar{g}(n)$, where $\bar{g}(n) = O(1)$

- **A003314 (binary entropy function):** $g(n) = n$ **with** $f(1) = 0$

$$f(n) = nL_n + 2n - 2^{L_n+1} = n\log_2 n + nP(\log_2 n),$$

where $P(t) := 2 - \{t\} - 2^{1-\{t\}}$. **A003314**$(n)$ **= A033156**$(n) - n$

- **A001855 (worst case of mergesort):** $g(n) = n - 1$; **essentially A123753 (differing by** n**) & A097384 (diff. initial conditions).**

- **A000788 (best case of mergesort):** $g(n) = \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2} + \bar{g}(n)$, **where** $\bar{g}(n) = \frac{1}{2} - \left\{\frac{n}{2}\right\} = \frac{1}{2}\mathbf{1}_{n \text{ is even}}$.

$$\Longrightarrow f(n) = \frac{1}{2} n\log_2 n + nP(\log_2 n)$$

P **is the Trollope-Delange or Takagi function**

$$P(t) = \frac{1}{2} - \frac{1}{2}\{t\} - 2^{-\{t\}} + \sum_{k\geqslant 0} 2^{-k-\{t\}}\bar{g}(2^{k+\{t\}})$$

$$\bar{g}(x) = \begin{cases} \frac{1}{2}(1 - \{x\}) & \lfloor x \rfloor \text{ even} \\ \frac{1}{2}\{x\} & \lfloor x \rfloor \text{ odd} \end{cases}$$

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LINEAR $g(n)$

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$$\Longrightarrow f(n) = \frac{1}{2}n\log_2 n + nP(\log_2 n)$$

P **is the Trollope-Delange or Takagi function**

$$P(t) = \frac{1}{2} - \frac{1}{2}\{t\} - 2^{-\{t\}} + \sum_{k\geqslant 0} 2^{-k-\{t\}}\bar{g}(2^{k+\{t\}})$$

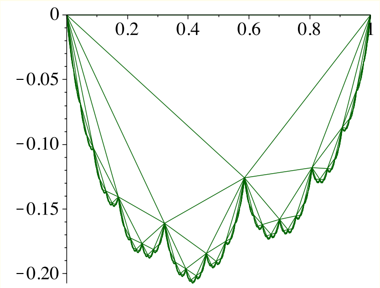
$$\bar{g}(x) = \begin{cases} \frac{1}{2}(1-\{x\}) & \lfloor x \rfloor \text{ even} \\ \frac{1}{2}\{x\} & \lfloor x \rfloor \text{ odd} \end{cases}$$

A000788 $\Lambda[f] = g$ WITH $g(n) = \lfloor \frac{n}{2} \rfloor$

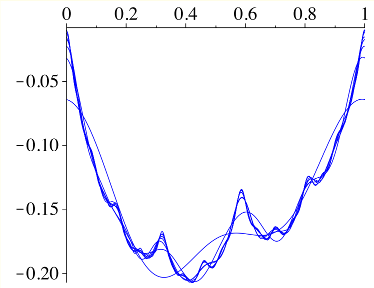
Same sequence also arises in

- **The number of bisecting strategies in certain games (Gilbert 1962)**
- **Linear forms in number theory (Lindstrom 1965)**
- **Determinant of some matrix of order n (Clements 1965)**
- **Bounds for the number of edges in certain class of graphs (Graham 1970, Hart 1976a)**
- **The solution to the recurrence** $f(n) = \max_k\{f(k) + f(n-k) + \min\{k, n-k\}\}$ **with** $f(1) = 0$ **is exactly** $S(n)$**; (Greene & Knuth 2008, McIlroy 1974, Hart 1976)**
- **The number of comparators used by Batcher's bitonic sorting network (Hong & Sedgewick 1982)**
- **External left length of some binary trees (Li 1986)**
- **#(1's) in the binary code of the 1st** $n-1$ **pos. int. (Delange 1975)**
- **The minimum number of comparisons used by**
  - **top-down recursive mergesort (Flajolet & Golin, 1994)**
  - **bottom-up mergesort (Panny & Prodinger 1995)**
  - **queue-mergesort (Chen et al. 1999)**

A000788: BEST CASE OF MERGESORT



$P(\log_2 n)$ **by** $\frac{f(n)}{n} - \frac{1}{2}\log_2 n$



$P(t)$ **by Fourier**

Takagi (1901) function: $\sum_{k\geqslant 0} 2^{-k-\{t\}}\bar{g}(2^{k+\{t\}})$

One of the first C^0 but nowhere differentiable functions

Closely connected sequences: A078903 (diff. by $n-1$), A076179 ($=2\cdot$A078902), & A268289 ($\Lambda[f] = 2\bar{g}$)

- **Average case of mergesort (Flajolet & Golin, '94):**

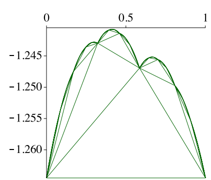
$$g(n) = n - \frac{\lfloor \frac{n}{2} \rfloor}{\lceil \frac{n}{2} \rceil + 1} - \frac{\lceil \frac{n}{2} \rceil}{\lfloor \frac{n}{2} \rfloor + 1} = n - 1 + \bar{g}(n)$$

$$\Longrightarrow f(n) = n \log_2 n + nP(\log_2 n) - \sum_{k \geqslant 0} \frac{1}{2^k(2^k n + 1)}$$

$$P(t) = 1 - \{t\} - 2^{1-\{t\}} + \sum_{k \in \mathbb{Z}} 2^{-k-\{t\}} \bar{g}(2^{k+\{t\}})$$

$$\widehat{P}(0) = \frac{1}{2} - \frac{1}{\log 2} - \frac{2}{\log 2} \sum_{m \geqslant 1} \frac{\log(2m+1) - \log(2m)}{(m+1)(m+2)}$$

$$\widehat{P}(k) = \frac{1}{\chi_k(\chi_k + 1)\log 2} \left(1 - 2 \sum_{m \geqslant 1} \frac{m^{-\chi_k} - (m + \frac{1}{2})^{-\chi_k}}{(m+1)(m+2)}\right)$$



Rederived results in Flajolet & Golin (1994), H. (1998)

LINEAR $g(n)$

- **Quicksort, best case (Sedgewick, 1975):**

$$a(n) = a\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) + a\left(\left\lceil \frac{n-1}{2} \right\rceil\right) + n - 1$$

equals A061168 (=A083652 shifting by n)

$$f(n) = a(n+1) \Longrightarrow \Lambda[f] = n-2 \Longrightarrow f(n) = \sum_{k<n} \lfloor \log_2 n \rfloor$$

$$\Longrightarrow f(n) = n \log_2 n + nP(\log_2 n) + 2$$

$$(P(t) = -\{t\} - 2^{1-\{t\}})$$

$a(n) = a\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) + a\left(\left\lceil \frac{n-1}{2} \right\rceil\right) + cn + d$ ***similarly treated***

- **A067699: #(cmps) made in a version of Quicksort:**

$g(n) = 2\left\lceil \frac{n+1}{2} \right\rceil \Longrightarrow f(n) = n \log_2 n + nP(\log_2 n) - 2$

$$P(t) = 4 - \{t\} - 2^{1-\{t\}} - \sum_{k \geqslant 1} 2^{-k-\{t\}} \bar{g}(2^{k+\{t\}})$$

$\bar{g}(x) = \{x\}$ **if** $\lfloor x \rfloor$ **even, &** $\bar{g}(x) = 1 - \{x\}$ **if** $\lfloor x \rfloor$ **odd**

LINEAR $g(n)$

- **Quicksort, best case (Sedgewick, 1975):**

$$a(n) = a\left(\left\lfloor \tfrac{n-1}{2} \right\rfloor\right) + a\left(\left\lceil \tfrac{n-1}{2} \right\rceil\right) + n - 1$$

equals A061168 (=A083652 shifting by n)

$$f(n) = a(n+1) \implies \Lambda[f] = n - 2 \implies f(n) = \sum_{k<n} \lfloor \log_2 n \rfloor$$

$$\implies f(n) = n \log_2 n + nP(\log_2 n) + 2$$

$$(P(t) = -\{t\} - 2^{1-\{t\}})$$

$a(n) = a\left(\left\lfloor \tfrac{n-1}{2} \right\rfloor\right) + a\left(\left\lceil \tfrac{n-1}{2} \right\rceil\right) + cn + d$ ***similarly treated***

- **A067699: #(cmps) made in a version of Quicksort:**

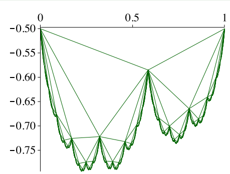
$g(n) = 2\left\lceil \tfrac{n+1}{2} \right\rceil \implies f(n) = n \log_2 n + nP(\log_2 n) - 2$

$$P(t) = 4 - \{t\} - 2^{1-\{t\}} - \sum_{k\geqslant 1} 2^{-k-\{t\}} \bar{g}(2^{k+\{t\}})$$

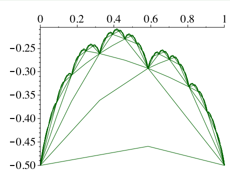
$\bar{g}(x) = \{x\}$ **if** $\lfloor x \rfloor$ **even, &** $\bar{g}(x) = 1 - \{x\}$ **if** $\lfloor x \rfloor$ **odd**

MORE LINEAR $g(n)$

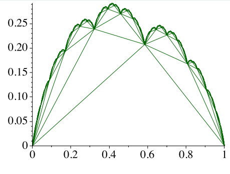
OEIS	description	$g(n)$	$f(n)$
A220001	**Benes n/w**	$2\left\lfloor \frac{n}{2} \right\rfloor (f(2)=1)$	$n\log_2 n + nP()$
–	**Merging n/w**	$\left\lfloor \frac{n}{2} \right\rfloor - \mathbf{1}_{n=4\ell}$	$\frac{1}{2} n\log_2 n + nP() + 1$
A173318	**Gray code**	$\left\lfloor \frac{n+1}{4} \right\rfloor + \left\lfloor \frac{n+2}{4} \right\rfloor$	$\frac{1}{2} n\log_2 n + nP()$
–	**Lebesgue const. (Walsh)**	$\frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor + \frac{1}{2}h(n) - \frac{1}{2}h(\lceil \frac{n}{2} \rceil)$ $(\frac{1}{2}\Lambda[h] = \mathbf{1}_{n\ \text{odd}})$	$\frac{1}{4} n\log_2 n + nP() - \frac{1}{2}h(n)$



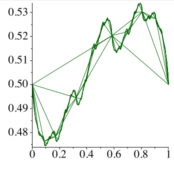
A220001



merging n/w



A173318



Lebesgues

$$f(n) = n + \min_{1 \leqslant k < n}\{f(k) + f(n-k)\} \ (n \geqslant 2)$$

- **A003314:** $f(1) = 0 \Longrightarrow k = \lfloor \frac{n}{2} \rfloor$ **opt.**
- **A033156:** $f(1) = 1 \Longrightarrow k = \lfloor \frac{n}{2} \rfloor$ **opt.**
- **A054248:** $f(1) = 1, f(2) = 2 \Longrightarrow \boxed{k = 2\lfloor \frac{n+2}{4} \rfloor}$ **opt.**
  $\Lambda_{1,1}[f] = g$ **with** $g(n) = n - 2 \cdot \mathbf{1}_{n \equiv 2 \bmod 4}$ **&** $f(1) = 1$

$$f(n) = n(L_n + 3) - 2^{L_n+1} - 2\lfloor \tfrac{n}{2} \rfloor$$

$$f(n) = n + \min_{1 \leqslant k < n}\{f(k) + f(n-1-k)\} \ (n \geqslant 2)$$

- **A001855 (worst-case mergesort):** $f(1) = 1$
- **A097383:** $f(1) = 0 \Longrightarrow \boxed{k = 2\lfloor \frac{n+2}{4} \rfloor}$ **opt.**
  $\Lambda_{1,1}[f] = g$ **with** $g(n) = n - 1 - \mathbf{1}_{n \equiv 2 \bmod 4}$ **&** $f(1) = 0$

$$f(n) = n(L_n + 1) - 2^{L_n+1} - \lfloor \tfrac{n}{2} \rfloor + 1$$

QUADRATIC $g(n)$

- **A122247 ($f(n)$ = partial sum of $a(n) = a(\lfloor \frac{n}{2} \rfloor) + \lfloor \frac{n}{2} \rfloor$):** $g(n) = \frac{n^2}{4} - \frac{n}{2} + 1 + \frac{1}{2}\{\frac{n}{2}\}$. **Define** $\bar{f}(n) := \frac{n^2}{2} - f(n) - 1$. **Then** $\Lambda[\bar{f}] = \bar{g}$ **with** $\bar{g}(n) = \lfloor \frac{n}{2} \rfloor$ **&** $\bar{f}(1) = -\frac{1}{2}$.

$$\Longrightarrow f(n) = \frac{1}{2}n^2 - \frac{1}{2}n\log_2 n + nP(\log_2 n) - 1,$$

where $P(t)$ = **Trollope-Delange shifted by** $-\frac{1}{2}$.

- **A077071:** $g(n) = n^2 - n$.

$$f(n) = \sum_{0 \leqslant k < n} v_2\left(\text{denominator}\left(4^{n-1}\binom{2n-2}{2k}\binom{n+k-\frac{3}{2}}{2n-2}\right)\right),$$

($v_2(n)$ = **largest power of two dividing** n)

$$f(n) = 2n^2 - n\log_2 n - 2nP(\log_2 n),$$

where P = **Trollope-Delange shifted by** 1.

$g(n) = n^2 \Longrightarrow f(n) = 2n^2 + nP(\log_2 n)$ $\quad$ ($P \propto f(1)$)

- **A001105:** $\begin{cases} g(n) = n^2 - \mathbf{1}_{n \text{ is odd}} \\ f(1) = 0 \end{cases} \Longrightarrow \boxed{f(n) = 2n(n-1)}$

  $\boxed{\textbf{\textit{fluctuation on }} f(n) \longrightarrow \textbf{\textit{fluctuation on }} g(n)}$

$$g(n) = n^2 - \mathbf{1}_{n \text{ is odd}} \Longrightarrow \boxed{f(n) = 2n(n-1) + f(1)n}$$

$\Longrightarrow$ **Many OEIS sequences**

A046092, A000217, A002378

A005563, A001844, A016744

A161680, $\cdots$

$$g(n) = \frac{3}{4}n^3 - \frac{3}{2}n\left\{\frac{n}{2}\right\} \Longrightarrow f(n) = n^3 \ \textbf{(A000578)}$$

$$g(n) = \begin{cases} (1 - 2^{1-m})n^m, & n \textbf{ even} \\ (1 - 2^{1-m})n^m - \displaystyle\sum_{1 \leqslant j \leqslant \lfloor \frac{m}{2} \rfloor} \frac{\binom{m}{2j}}{2^{m-1}} n^{m-2j}, & n \textbf{ odd} \end{cases} \Longrightarrow \boxed{f(n) = n^m}$$

QUADRATIC & HIGHER ORDER $g(n)$

- **A001105:** $\begin{cases} g(n) = n^2 - \mathbf{1}_{n \text{ is odd}} \\ f(1) = 0 \end{cases} \Longrightarrow \boxed{f(n) = 2n(n-1)}$

  fluctuation on $f(n) \longrightarrow$ ***fluctuation on*** $g(n)$

$$g(n) = n^2 - \mathbf{1}_{n \text{ is odd}} \Longrightarrow \boxed{f(n) = 2n(n-1) + f(1)n}$$

$\Longrightarrow$ **Many OEIS sequences**

A046092, A000217, A002378

A005563, A001844, A016744

A161680, $\cdots$

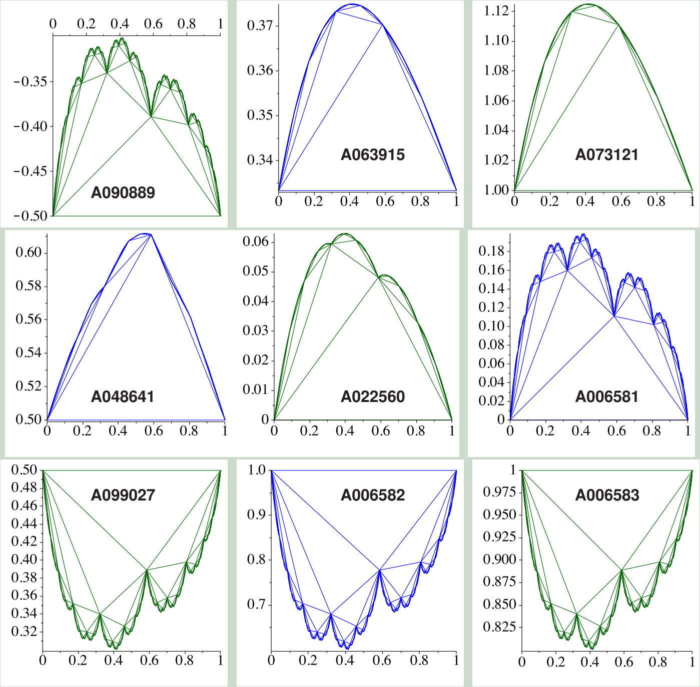
$$g(n) = \tfrac{3}{4}n^3 - \tfrac{3}{2}n\left\{\tfrac{n}{2}\right\} \Longrightarrow f(n) = n^3 \text{ (A000578)}$$

$$g(n) = \begin{cases} (1-2^{1-m})n^m, & n \text{ even} \\ (1-2^{1-m})n^m - \displaystyle\sum_{1 \leqslant j \leqslant \lfloor \frac{m}{2} \rfloor} \frac{\binom{m}{2j}}{2^{m-1}} n^{m-2j}, & n \text{ odd} \end{cases} \Longrightarrow \boxed{f(n) = n^m}$$

APPLICABLE TO $f(n) = \alpha f(\lfloor \frac{n}{2} \rfloor) + \alpha f(\lceil \frac{n}{2} \rceil) + g(n)$

OEIS examples: $\boxed{\alpha = 2}$

OEIS	description	$g(n)$	$f(n)$
A090889	**Double partial sum of the dyadic valuation** $v_2(n)$ $\sum_{1\leqslant k<n} kv_2(k)(n-k)$	$\begin{cases} \frac{n(n^2-4)}{12} & \text{if } n \text{ is even} \\ \frac{n(n^2-1)}{12} & \text{if } n \text{ is odd} \end{cases}$	$\frac{n^3}{6} + n^2 P(\log_2 n) + \frac{n}{3}$
A063915	$\sum_{1\leqslant j<n} 2^{\lfloor j}$	1	$n^2 P(\log_2 n) - \frac{1}{3}$
A073121	**Analysis of a regular expression algorithm**	0	$n^2 P(\log_2 n)$ $2^{-\{t\}}(3-2^{1-\{t\}})$
A048641	$\sum_{j<n}$ **Gray-code-function**(j) **(decimal equiv. of Gray code)**	$\lfloor \frac{n}{2} \rfloor + \mathbf{1}_{n \bmod 4 \equiv 3}$	$n^2 P(\log_2 n) - \frac{n}{2}$
A022560	$\sum_{1\leqslant j<n} 2^{v_2(j)}(n-j)$ **highest power of 2 dividing** n	$\frac{n^2}{4} - \frac{1}{4}\mathbf{1}_{n \text{ is odd}}$	$\frac{n^2}{4}\log_2 n + n^2 P(\log_2 n)$
A006581	$\sum_{j<n}(j$ **AND** $(n-j))$ **AND: the bitwise and**	$2\{\frac{n}{2}\}\lfloor \frac{n}{2} \rfloor$	$n^2 P(\log_2 n)$
A099027	$\sum_{j<n}(\bar{j}$ **AND** $(n-j))$	$\begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$	$n^2 P(\log_2 n) - \frac{n}{2}$
A006582	$\sum_{j<n}(j$ **XOR** $(n-j))$	$\begin{cases} 3n-6 & \text{if } n \text{ is even} \\ 2n-6 & \text{if } n \text{ is odd} \end{cases}$	$n^2 P(\log_2 n) - 3n + 2$
A006583	$\sum_{j<n}(j$ **OR** $(n-j))$	$\begin{cases} 3n-6 & \text{if } n \text{ is even} \\ \frac{5}{2}n - \frac{13}{2} & \text{if } n \text{ is odd} \end{cases}$	$n^2 P(\log_2 n) - 3n + 2$


A090889
A063915
A073121
A048641
A022560
A006581
A099027
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A006583

A SUMMARY

$$f(n) = f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + f\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n)$$

$$\Longrightarrow f(n) \equiv F(n) + nP(\log_2 n) - G(n)$$

$$f(n) = \alpha f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \alpha f\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n)$$

$$\Longrightarrow f(n) \equiv F(n) + n^{1+\log_2 \alpha} P(\log_2 n) - G(n)$$

$F(n)$ large, $G(n)$ small

MORE GENERAL RECURRENCE

$$f(n) = \alpha f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \beta f\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n)?$$

$$\begin{cases} f(2n) = (\alpha + \beta) f(n) + g(2n) \\ f(2n+1) = \alpha f(n) + \beta f(n+1) + g(2n+1) \end{cases}$$

$$f(x) := f(\lfloor x \rfloor) + \varphi(\{x\})\big(f(\lfloor x \rfloor + 1) - f(\lfloor x \rfloor)\big) \qquad (x \geqslant 1)$$

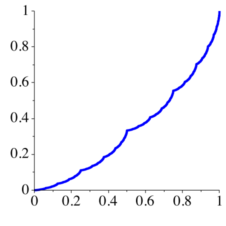
$$\Longrightarrow f(x) = (\alpha + \beta) f\left(\frac{x}{2}\right) + g(x) \qquad (x \geqslant 2)$$

THE INTERPOLATION FUNCTION φ

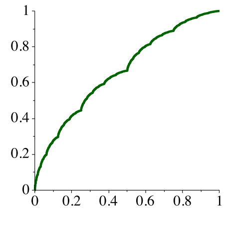
$$\varphi(0) = 0, \ \varphi(1) = 1$$

$$\varphi(t) = \begin{cases} \frac{\beta}{\alpha+\beta}\,\varphi(2t), & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ \frac{\alpha}{\alpha+\beta}\,\varphi(2t-1) + \frac{\beta}{\alpha+\beta}, & \text{if } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

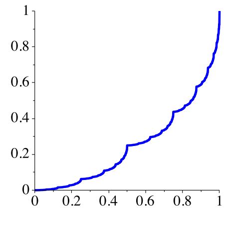
$$\varphi\left(t = \sum_{k\geqslant 1} 2^{-e_k}\right) := \sum_{k\geqslant 1} \alpha^{k-1}\beta^{e_k-k+1}(\alpha+\beta)^{-e_k} \uparrow$$ **& continuous**



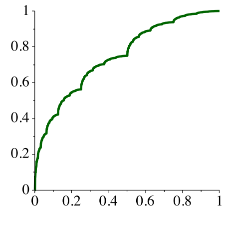
$(\alpha, \beta) = (2, 1)$



$(\alpha, \beta) = (1, 2)$



$(\alpha, \beta) = (3, 1)$



$(\alpha, \beta) = (1, 3)$

NEW RESULTS: $\alpha, \beta > 0$

Extend $g(x) := g(\lfloor x \rfloor) + \varphi(\{x\})\big(g(\lfloor x \rfloor + 1) - g(\lfloor x \rfloor)\big)$

The following statements are equivalent: $\varrho := \log_2(\alpha + \beta)$

- $f(n) = n^\varrho P(\log_2 n) + o(n)$ **as** $n \to \infty$**, for some continuous & 1-periodic function** P **on** $\in \mathbb{R}$.
- $G_k(t) := \sum_{1 \leqslant j \leqslant k} 2^{-j\varrho} g(2^j t)$ ***converges uniformly*** **for** $t \in [1, 2]$ **as** $k \to \infty$.

When both hold, the 1-periodic function P **is given by**

$$P(t) := \sum_{k \in \mathbb{Z}} \frac{g(2^{k+\{t\}})}{2^{(k+\{t\})\varrho}} + f(1)(\alpha + \beta)^{-\{t\}}\big(1 + (\alpha + \beta - 1)\varphi\big(2^{\{t\}} - 1\big)\big).$$

Furthermore, $\boxed{f(n) \equiv n^\varrho P(\log_2 n) - G(n)}$ **(an identity).**

$$g(n) = O\left(\frac{n^\varrho}{(\log n)^{1+\varepsilon}}\right) \Longrightarrow f(n) = n^\varrho P(\log_2 n) - \sum_{k \geqslant 1} 2^{-k\varrho} g(2^k n)$$

SMOOTHNESS OF P

A simple condition

$$\sum_{m\geqslant 1} 2^{m(1-\varrho)} \max_{2^m\leqslant k<2^{m+1}} |g(k+1)-g(k)| < \infty$$

$$\Longrightarrow \begin{cases} P \text{ has an abs. conv. Fourier series} \\ P \text{ is of bounded variation} \\ |P(x)-P(y)| \leqslant c|x-y|^{\log_2 \frac{\alpha+\beta}{\alpha\vee\beta}} \\ \qquad \text{(Hölder continuity)} \end{cases}$$

Fourier series of P: $P(t) = \sum_{k\in\mathbb{Z}} \widehat{P}(k) e^{2k\pi it}$

$$\widehat{P}(k) = \frac{1}{\log 2}\int_1^\infty \frac{g(u)}{u^{\varrho+\chi_k+1}}\,\mathrm{d}u + \frac{f(1)}{\log 2}\int_0^1 \frac{1+(\alpha+\beta-1)\varphi(u)}{(1+u)^{\varrho+\chi_k+1}}\,\mathrm{d}u$$

Applications

$$(\alpha, \beta > 0)$$

$$(\alpha \neq \beta)$$

SUM-OF-DIGITS FUNC.: GENE. POLYs

$$\Lambda_{\alpha,\beta}[f](n) := f(n) - \alpha f\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) - \beta f\left(\left\lceil \tfrac{n}{2} \right\rceil\right)$$

$f(n) = \sum_{k<n} \alpha^{\nu(k)}$ satisfies $\boxed{\Lambda_{\alpha,1}[f] = 0}$ with $f(1) = 1$

- **Stein (1986), Okada et al. (1995, 1996), Muramoto et al. (2000), Krüppel (2009)**

$$f\left(n = \sum_{1\leqslant j\leqslant s} 2^{e_j}\right) = \sum_{1\leqslant j\leqslant s} \alpha^{j-1}(\alpha+1)^{e_j} = n^{\log_2(\alpha+1)} P_\alpha(\log_2 n)$$

- **Grabner & H. (2005): Fourier series by analytic approach, which is absolutely convergent for** $\boxed{\alpha \in (\sqrt{2}-1, \sqrt{2}+1)}$

New: Fourier series of P_α converges absolutely for $\alpha > 0$

Zygmund's Theorem: ***The Fourier series of a function that is both of bounded variation & Hölder continuous with exponent*** $\lambda > 0$ ***in the unit interval is absolutely convergent.***

RECURRENCES WITH MIN OR MAX

$$\boxed{a_n = \min_{1\leqslant k\leqslant \lfloor \frac{n}{2}\rfloor} \{\alpha a_k + \beta a_{n-k}\}} \ (n \geqslant 2) \text{ with } a_1 = 1$$

Chang & Tsai (2000) (AND-OR Gates Problem):

$\beta \geqslant \alpha > 0 \Longrightarrow a_n = f(n)$, **where** $\Lambda_{\alpha,\beta}[f] = 0$ **w** $f(1) = 1$

$$\boxed{a_n = \max_{1\leqslant k\leqslant \lfloor \frac{n}{2}\rfloor} \{\alpha a_k + \beta a_{n-k}\}} \ (n \geqslant 2) \text{ with } a_1 = 1$$

$\alpha \geqslant \beta > 0 \Longrightarrow a_n = f(n)$, **where** $\Lambda_{\alpha,\beta}[f] = 0$ **w** $f(1) = 1$

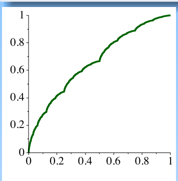
A064194 $(1,2)$	**A268524** $(1,3)$	**A268527** $(1,4)$	**A073121** $(2,2)$	**A268526** $(2,3)$
A006046 $(2,1)$	**A130665** $(3,1)$	**A116520** $(4,1)$	**A130667** $(5,1)$	**A268525** $(3,2)$

$$f(n) = n^{\log_2(\alpha+\beta)} P(\log_2 n)$$

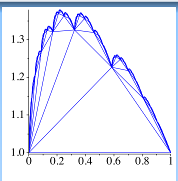
$f(n) = f(\lfloor \frac{n}{2} \rfloor) + 2f(\lceil \frac{n}{2} \rceil) + g(n)$

OEIS	description	$g(n)$	$f(1)$	$f(n)$ $P(t)$
A064194	**Karatsuba multiplication**	0	1	$f(n) = n^{\log_2 3} P(\log_2 n)$ $P(t) = 3^{-\{t\}}(1 + 2\varphi(2^{\{t\}} - 1))$
A268514	$\sum_{1 \leqslant k < n} 2^{\nu_0(k)}$	1	0	$f(n) = n^{\log_2 3} P(\log_2 n) - \frac{1}{2}$ $P(t) = \frac{1}{2} 3^{-\{t\}}(1 + 2\varphi(2^{\{t\}} - 1))$
A086845	**Bose-Nelson networks**	$\lfloor \frac{n}{2} \rfloor$	0	$f(n) = n^{\log_2 3} P(\log_2 n) - n$ $P(t) = 3^{-\{t\}}(1 + 2\varphi(2^{\{t\}} - 1))$ $+ \sum_{k \geqslant 0} \frac{\bar{g}(2^{k+\{t\}})}{3^{k+\{t\}}}$

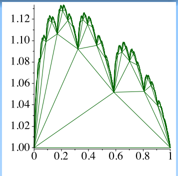
- $\Lambda_{1,2}[f] = \lceil \frac{n}{2} \rceil$ **(**$f(1) = 0$**):** $f(n) + n =$ **A064194**(n)
- **A080572:** $\sum_{0 \leqslant j,k < n} 1_{j \text{ AND } j \neq 0} \Longrightarrow n^2 -$ **A080572**$(n) =$ **A064194**(n)



$(1, 2) : \varphi(t)$



A064194



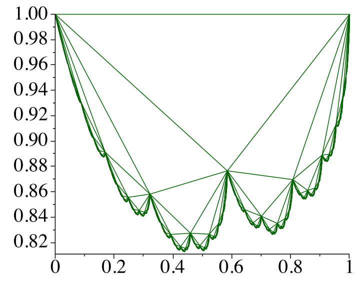
A086845

A006046: odd entries in Pascal triangle

An old problem (since 19-th century) with a rich literature

$f(n) := \sum_{0 \leqslant k < n} 2^{\nu(k)}$ **satisfies** $\Lambda_{2,1}[f] = 0$ **with** $f(1) = 1$

- **A051679:** $\binom{n+1}{2} - f(n)$
- **A074330:** $f(n+1) - 1$
- **A116593:** $f(n) + f(n+2)$
- **A151566:** $2f(n) + \cdots$
- **A160722:** $3f(n) - 2n$
- **A171378:** $(n+1)^2 - f(n)$
- **A245550:** $f(n) - 2f(\lfloor \frac{n-1}{2} \rfloor)$
- **A256256:** $6f(n)$



- **A193494 (a binary tree algo):** $f(n) = a(n-1)$, **where** $a(n) = 1 + \max_{0 \leqslant k < n/2} (2a(k) + a(n-1-k))$

$$f(n) = \tfrac{1}{2}\textbf{A006046}(n) - \tfrac{1}{2}$$

$\beta = 0$: $f(n) = \alpha f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + g(n)$

$f(n) \quad n \in \mathbb{N}^+ \mapsto f(\lfloor x \rfloor) \quad x \in \mathbb{R}$

The following statements are equivalent: $\varrho := \log_2 \alpha$

- $n^{-\varrho} f(n) = P(\log_2 n) + o(1)$ **as** $n \to \infty$, **for some 1-periodic function** P **on** $\in \mathbb{R}$ **satisfying** $|P(\log_2 x) - P(\log_2 \lfloor x \rfloor)| \to 0$ **as** $x \to \infty$.
- $G_k(t) := \sum_{1 \leqslant j \leqslant k} 2^{-j\varrho} g(\lfloor 2^j t \rfloor)$ ***converges uniformly*** **for** $t \in [1, 2]$ **as** $k \to \infty$.

When both hold, the 1-periodic function P **is given by**

$$P(t) := \sum_{k \in \mathbb{Z}} 2^{-(k+\{t\})\varrho} g\left(\left\lfloor 2^{k+\{t\}} \right\rfloor\right) + f(1)\alpha^{-\{t\}}.$$

Furthermore, $\boxed{f(n) \equiv n^{\varrho} P(\log_2 n) - G(n)}$.

P is discontinuous

AN EXAMPLE FOR $\beta = 0$

A038554: $\Lambda_{2,0}[f] = g$,

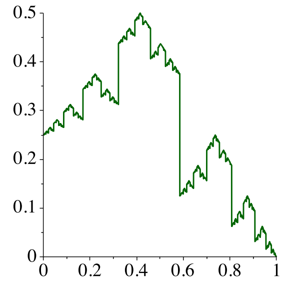
$$g(n) = \frac{1-(-1)^{\lceil \frac{n}{2} \rceil}}{2}$$

Connected to the XOR of n & its shift; also $f(n)$ equals Gray code function (A003188) minus 2^{L_n}.

$$f(n) = nP(\log_2 n)$$

$$P(t) := \sum_{k\in\mathbb{Z}} 2^{-k-\{t\}} g\left(\lfloor 2^{k+\{t\}} \rfloor\right)$$

P is discontinuous & $g(x) = 0$ for $x < 2$.



Many other examples: Ralf Stephan's OEIS webpage on divide-and-conquer sequences (2004).

$\alpha = 0$: $f(n) = \beta f\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n)$

$f(n) \quad n \in \mathbb{N}^+ \mapsto f(\lceil x \rceil) \qquad x \in \mathbb{R}$

The following statements are equivalent: $\varrho := \log_2 \alpha$

- $n^{-\varrho} f(n) = P(\log_2 n) + o(1)$ **as** $n \to \infty$**, for some 1-periodic function** P **on** $\in \mathbb{R}$ **satisfying** $|P(\log_2 x) - P(\log_2 \lceil x \rceil)| \to 0$ **as** $x \to \infty$**.**
- $G_k(t) := \sum_{1 \leqslant j \leqslant k} 2^{-j\varrho} g(\lceil 2^j t \rceil)$ ***converges uniformly*** **for** $t \in [1, 2]$ **as** $k \to \infty$**.**

When both hold, the 1-periodic function P **is given by**

$$P(t) := \sum_{k \in \mathbb{Z}} 2^{-(k+\{t\})\varrho} g\left(\left\lceil 2^{k+\{t\}} \right\rceil\right) + f(1)\beta^{1-\{t\}}.$$

Furthermore, $\boxed{f(n) \equiv n^{\varrho} P(\log_2 n) - G(n)}$**.**

P is discontinuous

Extensions & Variants

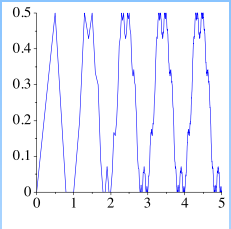
$\alpha, \beta < 0$: $f(n) = -f(\lfloor \frac{n}{2} \rfloor) - f(\lceil \frac{n}{2} \rceil) + g(n)$

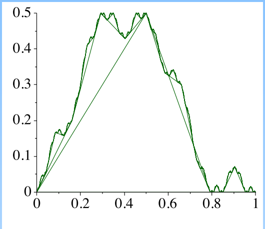
A0505536 (a von Koch sequence): $\Lambda_{-1,-1}[f] = g$, $g(n) = \lfloor \frac{n}{2} \rfloor$ with $f(0) = f(1) = 0$

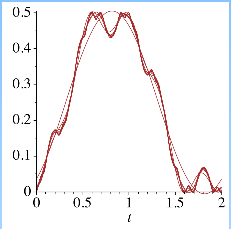
$\boxed{f(n) = nP(\log_2 n)}$, $P(t)$: C^0 & $\boxed{2\text{-periodic}}$

$$P(t) = \frac{1}{4} + \frac{(-1)^{\lfloor t \rfloor}}{2}\left(\frac{1}{2} - \frac{2^{1-\{t\}}}{3}\right) + \sum_{j \geqslant 0} (-1)^{j+\lfloor t \rfloor} \frac{\bar{g}(2^{j+\{t\}})}{2^{j+\{t\}}}$$

$$= \frac{1}{4} + \frac{3}{\log 2} \sum_{k \in \mathbb{R}} \frac{\zeta(\chi_k)}{\chi_k(\chi_k + 1)} e^{(2k+1)\pi i t} \qquad \left(\chi_k := \frac{(2k+1)\pi i}{\log 2}\right)$$







Another ex. A087733: $\Lambda_{-1,-1}[f] = g$, $g(n) = \frac{n^2}{4} - \frac{1}{2}\{\frac{n}{2}\}$.

BINARY TO q-ARY: $f(n) = \sum_{0\leqslant j<q} f\left(\left\lfloor \frac{n+j}{q} \right\rfloor\right) + g(n)$

$$f(n) = q\left(1 - \left\{\tfrac{n}{q}\right\}\right) f\left(\left\lfloor \tfrac{n}{q} \right\rfloor\right) + q\left\{\tfrac{n}{q}\right\} f\left(\left\lceil \tfrac{n}{q} \right\rceil\right) + g(n)$$

Same linear interpolation $\varphi(t) = t$ works well.

- **Oh & Kieffer (2010) (lossless compression of trees):**
  $g(n) = \log_2 \binom{q}{j \bmod q} \Longrightarrow f(n) = nP(\log_q n)$, $P(t) = \sum_{k\geqslant 1} \frac{g(q^{k+\{t\}})}{q^{k+\{t\}}}$

$$P(t) = \frac{D_q'(0)}{\log q} + \frac{1}{\log q} \sum_{k\neq 0} \frac{D_q(\chi_k)}{\chi_k(\chi_k+1)} e^{2k\pi it}$$

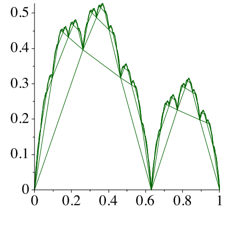
$$D_q(s) = \lambda(1)q^{-s}(2\zeta(s) - 1) + \sum_{2\leqslant j<q} \Delta^2\lambda(j-1)\left(q^{-s}\zeta\left(s, \tfrac{j}{q}\right) - j^{-s}\right)$$

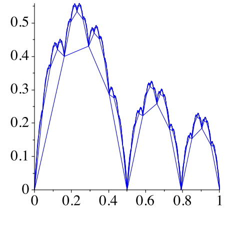
- **Sum-of-digits fun.:** $g(n) = \sum_{0\leqslant j<q}(q-1-j)\left\lfloor \frac{n+j}{q} \right\rfloor$

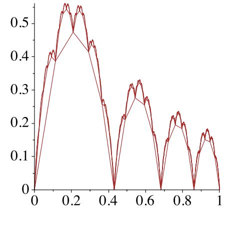
q	OEIS	q	OEIS	q	OEIS
2	A000788	5	A231668	8	A231680
3	A094345	6	A231672	9	A231684
4	A231664	7	A231676	10	A037123

$$f(n) = q\left(1-\left\{\frac{n}{q}\right\}\right) f\left(\left\lfloor \frac{n}{q}\right\rfloor\right) + q\left\{\frac{n}{q}\right\} f\left(\left\lceil \frac{n}{q}\right\rceil\right) + g(n)$$

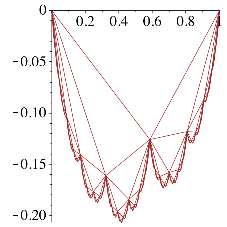
$$g(n) = \log_2 \binom{q}{j \bmod q} \Longrightarrow f(n) = nP(\log_q n),\ q = 3, 4, 5$$



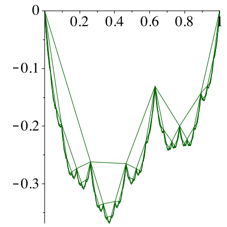




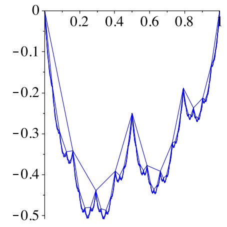
$$g(n) = \sum_{0\leqslant j<q}(q-1-j)\left\lfloor\frac{n+j}{q}\right\rfloor \Longrightarrow f(n) = \frac{q-1}{2} n\log_q n + nP(\log_q n)$$



$q = 2$



$q = 3$

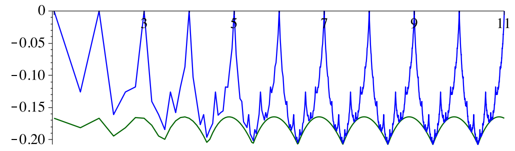


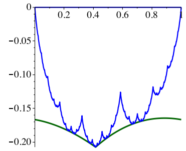
$q = 4$

SENSITIVITY

Compare $\Lambda[f](n) = \lfloor \frac{n}{2} \rfloor$ (best case of mergesort) with

$$\tilde{f}(n) = \begin{cases} \tilde{f}(\lfloor \frac{n}{2} \rfloor) + \tilde{f}(\lceil \frac{n}{2} \rceil) + \lfloor \frac{n}{2} \rfloor, & \textbf{if } n \not\equiv 0 \bmod 4 \ \& \ n \geqslant 2 \\ \tilde{f}(\frac{n}{2} - 1) + \tilde{f}(\frac{n}{2} + 1) + \frac{n}{2}, & \textbf{if } n \equiv 0 \bmod 4 \ \& \ n \geqslant 4 \end{cases}$$





$\tilde{f}(n) \leqslant f(n)$ ***for all*** $n \geqslant 1$

$$\tilde{f}(n+1) - \tilde{f}(n-1) = \lfloor \log_2 3n \rfloor$$

$$\tilde{f}(n) = \frac{n}{2}\log_2 n + nP(\log_2 n) + \begin{cases} \frac{1}{2} - \frac{(-1)^{\lfloor \log_2 3j \rfloor}}{6}, & \textbf{if } n \textbf{ is even}; \\ \frac{1}{4} + \frac{(-1)^{\lfloor \log_2 3j \rfloor}}{12}, & \textbf{if } n \textbf{ is odd}, \end{cases}$$

$$P(t) = \log_2 3 - \tfrac{1}{2}\{t + \log_2 3\} - 2^{-\{t + \log_2 3\}}$$

MANY OTHER EXTENSIONS

- **Large toll functions**
- **non-differentiability & other smoothness properties can be further characterized**
- **Connection between g and amplitude & shape of P?**
- **When P reaches min & max? algorithmic interpretation?**
- **A237686 Nim fractals (& many other variants):**

$$\begin{cases} f(2n) = f(n) + 7f(n-1) \\ f(2n+1) = 7f(n) + f(n+1) \end{cases}$$

- **Random tries: similar functional equations; $\frac{\log p}{\log q} \in \mathbb{Q}$ clarified, but irrational case still less clear**

Two common types of periodic functions:

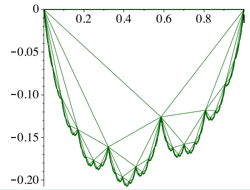
C^0 & *nowhere differentiable*

- **best case of mergesort**

$$f(n) = f\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) + f\left(\left\lceil \tfrac{n}{2} \right\rceil\right) + \left\lfloor \tfrac{n}{2} \right\rfloor$$

$$\Rightarrow f(x) = 2f\left(\tfrac{x}{2}\right) + \boxed{\begin{cases} \left\lfloor \tfrac{x}{2} \right\rfloor, & \lfloor x \rfloor \ \textbf{even} \\ \left\lfloor \tfrac{x}{2} \right\rfloor + \{x\}, & \lfloor x \rfloor \ \textbf{odd} \end{cases}}$$

$$n^{-1} f(n) = \tfrac{1}{2} \log_2 n + P(\log_2 n)$$



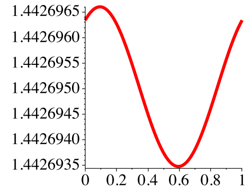
C^∞ with *minute amplitude*

- **expected size of a random n-trie**

$$f(z) = 2f\left(\tfrac{z}{2}\right) + \boxed{1 - (1+z)e^{-z}}$$

$$n^{-1}\mathbf{E}(\text{size of a random } n\text{-trie})$$

$$= \frac{1}{\log 2} + \underbrace{G(\log_2 n)}_{|\cdot| \leqslant 10^{-6}} + O(n^{-1})$$



AOFA: PERIODIC OSCILLATIONS EVERYWHERE

Two common types of periodic functions:

C^0 & *nowhere differentiable*

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$$f(n) = f\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) + f\left(\left\lceil \tfrac{n}{2} \right\rceil\right) + \left\lfloor \tfrac{n}{2} \right\rfloor$$

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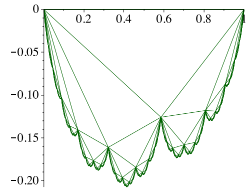

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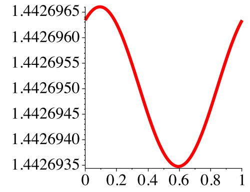

CONCLUSIONS

DaC RRs with balanced part sizes exhibited a surprisingly simple asymptotic pattern

- **a simple, effective approach**
- **identity in addition to asymptotic**
- **exponentially converging terms for periodic function**
- **smoothness characterized**
- **a large number of examples compiled and worked out**

A simple (missing) theory completed

- **approximating $\lfloor\cdot\rfloor$ or $\lceil\cdot\rceil$ by only one of these loses continuity of periodic functions and changes min/max of P $\Longrightarrow$ *Don't do that*!!**
- **minor modification of g may lead to drastic change of P**

THANK YOU