Bootstrap percolation on $G(n, p)$

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joint work with Mihyun Kang

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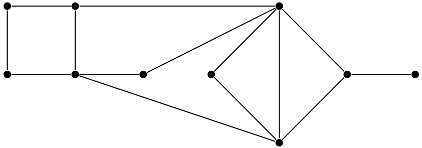

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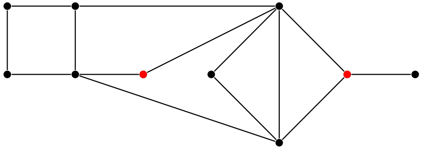

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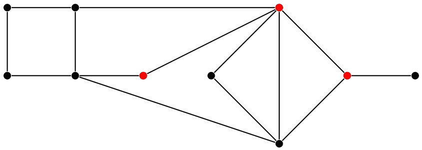

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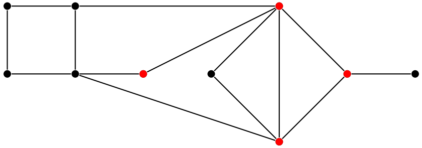

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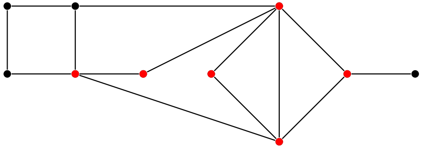

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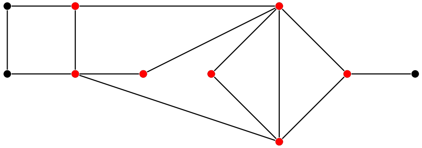

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- Initial infection set of size $a(n)$ uniformly at random
- Probability that almost every vertex becomes infected

Previous Results

Janson, Łuczak, Turova and Vallier (2012)

p	$\mathbb{P}$[almost infection]
$O(n^{-1})$	$o(1)$
$n^{-1} \ll p \ll n^{-1/r}$	$\mathbb{P}\left[N(0,1) \leq \frac{a-a_c}{\sqrt{a_c}}\right] + o(1)$
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Let $\omega_0 = \omega(\sqrt{a_c})$

- if $a \leq a_c - \omega_0$, then $\mathbb{P}$[almost infection]$= o(1)$
- if $a \geq a_c + \omega_0$, then $\mathbb{P}$[almost infection]$= 1 - o(1)$

Main Result (Subcritical)

Theorem Kang, M.

Set

- $t_0 = (r!/np^r)^{1/(r-1)}$
- $n^{-1} \ll p \ll n^{-1/r}$
- $\omega_0 = \omega(\sqrt{a_c})$

If $a = a_c - \omega_0$, then (for n large enough) with probability at least

$$1 - \exp\left(-\frac{\omega_0^2}{10 t_0}\right)$$

we have $|A_f| < t_0$.

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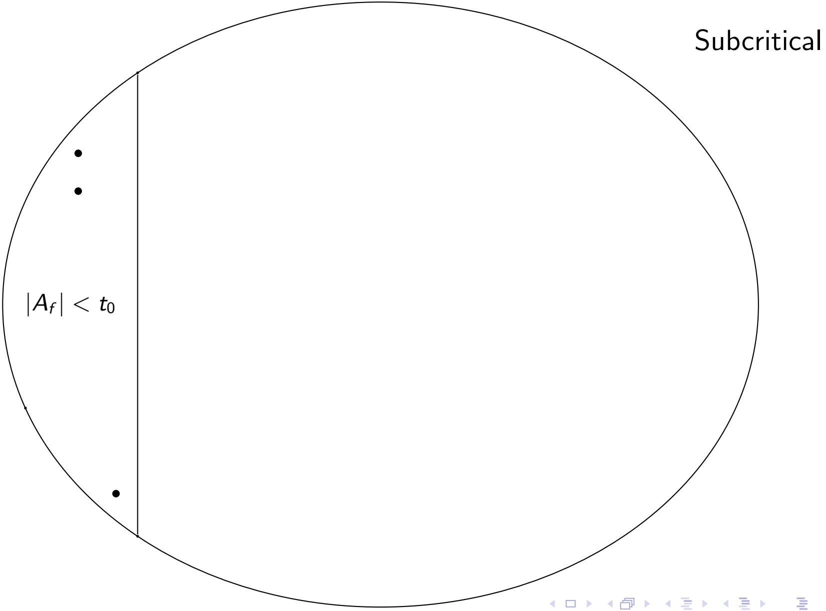
we have $|A_f| = (1 + o(1))n$.

Spread of infection



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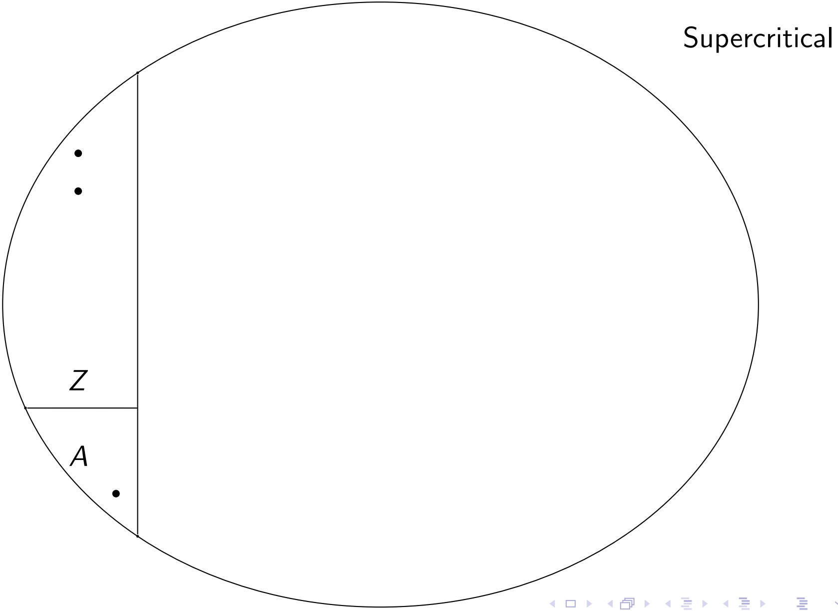

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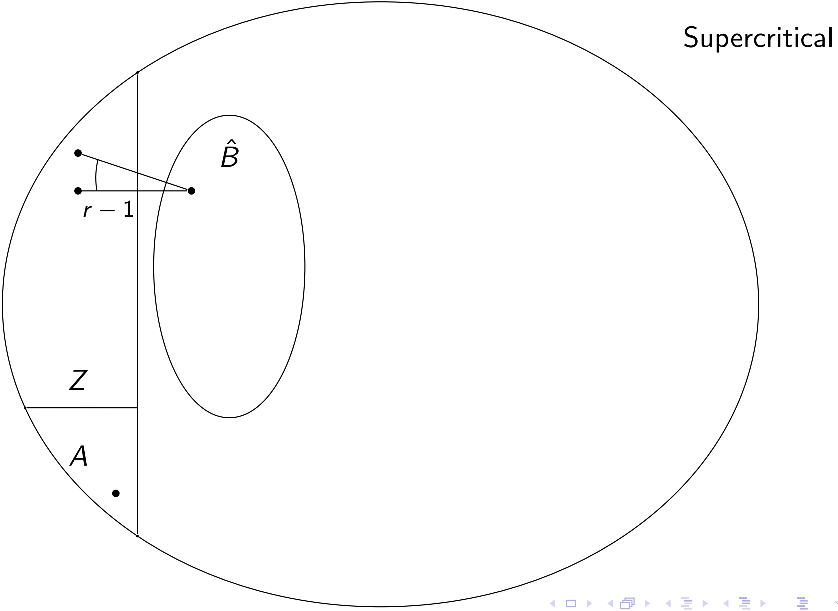

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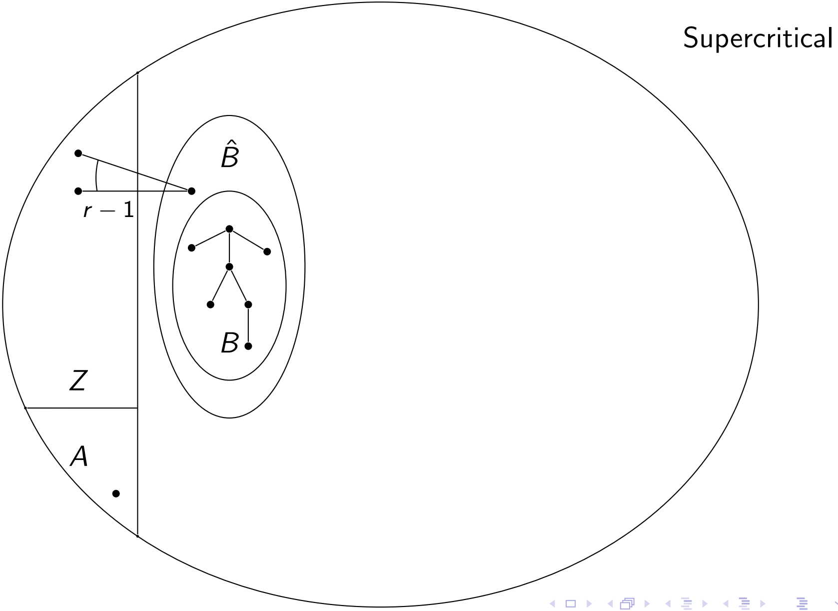

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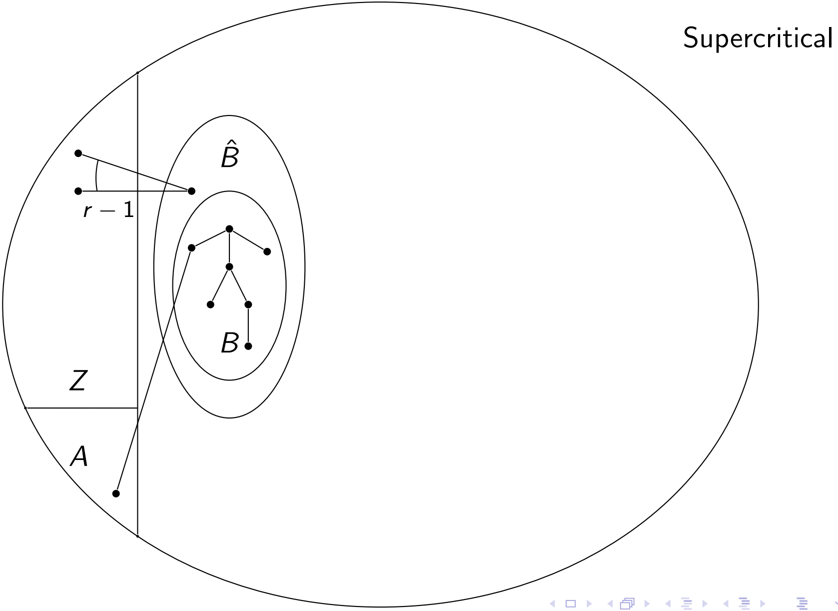

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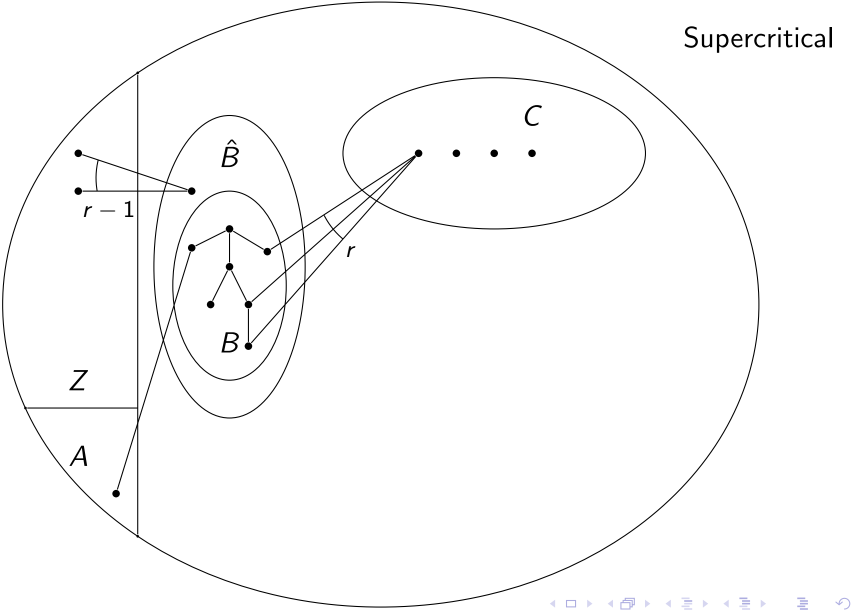

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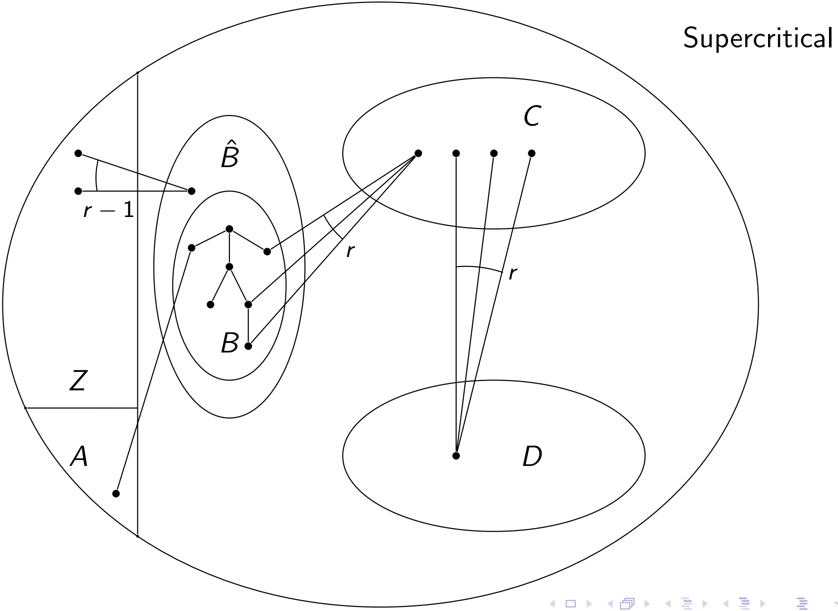

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No vertices are infected after $|A(t)| = |Z(t)| = t$

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- Need: tight concentration for every step

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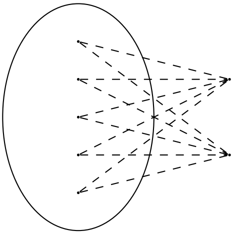

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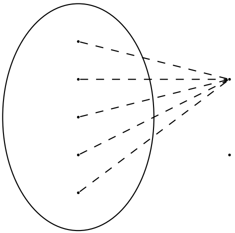

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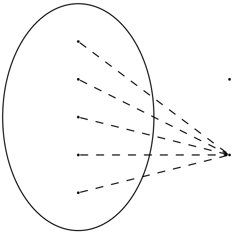

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- $A(t,i)$, Martingale $|A(t,i)|$

Concentration

Theorem — Chung, Lu (2006)

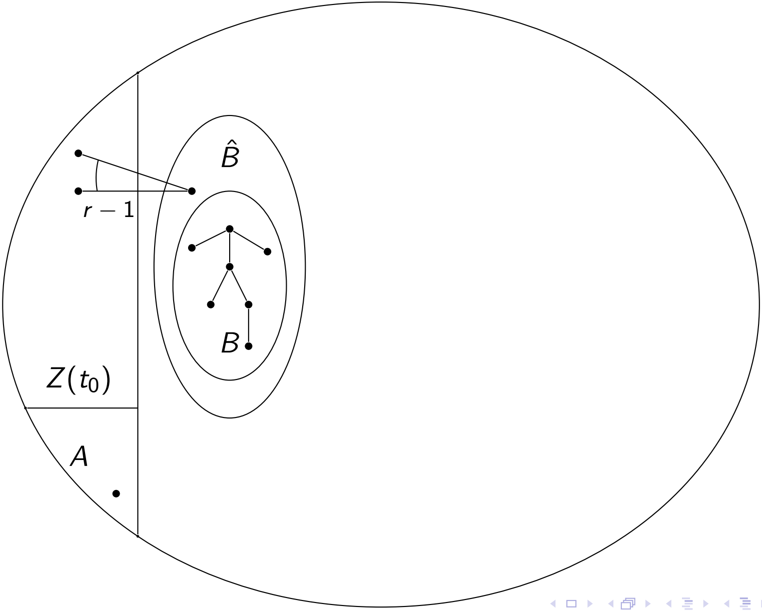
For $m_0 \in \mathbb{R}$ let $m_0, M_1, \ldots, M_k$ be a martingale whose conditional variance and differences satisfy the following: for each $1 \leq i \leq k$,

- $\mathrm{Var}[M_i | M_{i-1}, \ldots, M_0] \leq \sigma_i^2$;
- $|M_i - M_{i-1}| \leq m$ for some positive m.

Then for any $\lambda > 0$, we have

$$\mathbb{P}[M_k - m_0 \geq \lambda] \leq \exp\left(-\frac{\lambda^2}{2\left(\sum_{i=1}^k \sigma_i^2 + m\lambda/3\right)}\right).$$

Giant component



Giant component

Theorem Bollobás, Riordan (2016+)

Let $c > 1$, $\varepsilon > 0$ be constants independent of n. Then with probability

$$1 - \exp(-\Omega(n))$$

the binomial random graph $G(n, c/n)$ has a component of size at least $(1-\varepsilon)\rho n$, where $\rho \in (0, 1)$ is the unique positive solution of $1 - \rho = \exp(-c\rho)$.

Discussion

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- choice of t_0

Thank you!