Additive functionals of d-ary increasing trees

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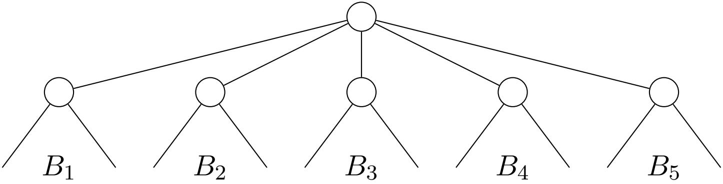
Additive functionals

Given a functional f of rooted trees, we say that a functional F is additive if it satisfies the recursion

$$F(T) = \sum_j F(B_j) + f(T),$$

where $B_1, B_2, \ldots$ are the branches of the root of T. The functional f is called the *toll function* of F.



Fringe subtrees

A *fringe subtree* of a rooted tree is a subtree induced by a vertex and all its descendants. If we denote by $\mathcal{F}(T)$ the collection of all fringe subtrees of T then the additive functional F can be expressed as follows:

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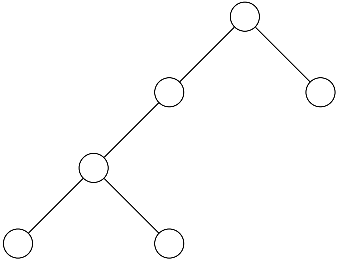

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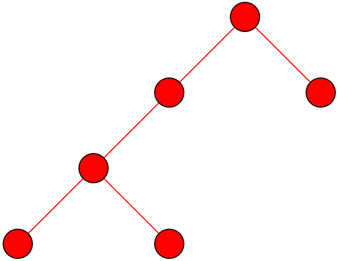

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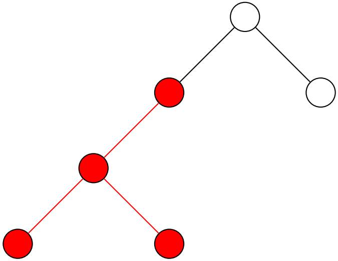

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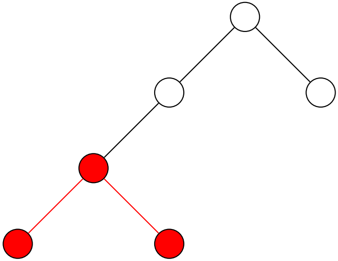

Additive functionals: examples

Here are some examples of additive tree functionals:

- The number of leaves: the toll function is

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- The logarithm of the size of the automorphism group: the toll function is

$$f(T) = \log R(T),$$

where $R(T)$ is the group of symmetries of the collection of root branches.

Motivation

Bóna and Flajolet studied symmetries in random phylogenetic trees (these are nonplane binary trees labelled at their leaves):

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Remark

The size of the automorphism group of a random phylogenetic tree on n labelled nodes is 2^{X_n}.

Uniform d-ary increasing tree

A uniform d-ary increasing tree can be obtained by the following growth process: starting with the root vertex carrying the label 1, we attach the vertex labelled n to one of the $(d-1)(n-1)+1$ places available in total uniformly at random.

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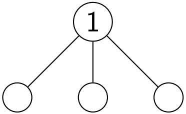

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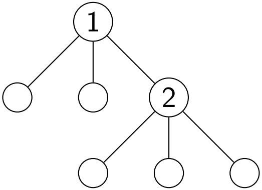

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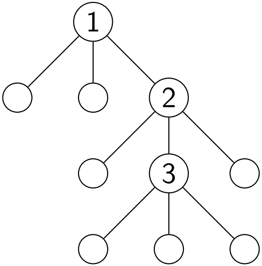

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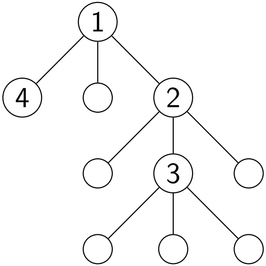

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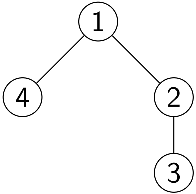

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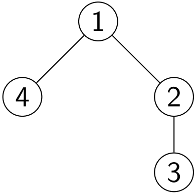

The number of d-ary increasing trees of order n is

$$Y_n = \prod_{j=1}^{n-1} ((d-1)j+1).$$

Our main result

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We consider toll functions that satisfy the following conditions:

(C1) f is bounded

(C2) $\mathbb{E}|f(T_n)| \to 0$ as $n \to \infty$

(C3) $\sum_{n\geq 1} \frac{\mathbb{E}|f(T_n)|}{n} < \infty$

Our main result

Theorem

If the toll function $f(T)$ satisfies (C1), (C2) and (C3), then there exist constants μ and σ such that the mean and variance of $F(T_n)$ are asymptotically

$$\mathbb{E}F(T_n) = \mu n + \frac{\mu}{d-1} + o(1), \quad \textbf{Var}F(T_n) = \sigma^2 n + o(n).$$

If $\sigma \neq 0$, then the renormalised random variable $(F(T_n) - \mu n)/\sqrt{\sigma^2 n}$ converges weakly to a standard normal distribution.

The constants

The constants μ and σ have the following expressions:

$$\mu = (d-1)\sum_{T} f(T) \prod_{j=1}^{|T|} \frac{1}{(d-1)j+d},$$

$$\sigma^2 = -\frac{\mu^2}{d-1} - (d-1)\sum_{T} \frac{f(T)^2 - 2f(T)(F(T)-\mu|T|)}{\prod_{j=1}^{|T|}((d-1)j+d)} +$$

$$d\sum_{T_1}\sum_{T_2} \frac{(d-1)^{1-|T_1|-|T_2|} f(T_1)f(T_2)}{(|T_1|-1)!(|T_2|-1)!} \int_0^1 \phi_{|T_1|}(x)\phi_{|T_2|}(x)dx,$$

where the sums are taken over all d-ary increasing trees and

$$\phi_k(x) = (1-x)^{-1}\int_x^1 (1-w)^{d/(d-1)} w^{k-1} dw.$$

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- Wagner (2015) obtained central limit theorems for additive functionals with bounded toll functions and exponentially small in average for simply generated families of trees, Pólya trees, recursive trees, and binary search trees.
- Holmgren and Janson (2014) for additive functionals in random binary search trees and random recursive trees.

Idea of the proof

We apply a truncation argument already used in Holmgren and Janson (2014).

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Lemma

If $(X_n)_{n\geq 1}$ *and* $(W_{m,n})_{m,n\geq 1}$ *are sequences of centred random variables such that*

- $W_{m,n} \xrightarrow{d}_n W_m$, *and* $W_m \xrightarrow{d}_m W$, *where* W *has a continuous distribution function,*
- $\mathrm{Var}(X_n - W_{m,n}) \to_n \gamma_m^2$ *and* $\gamma_m \to_m 0$,

then $X_n \xrightarrow{d}_n W$.

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For our purpose:

- $X_n = (F(T_n) - \mathbb{E}F(T_n))/n$
- $W_{n,m} = (F_m(T_n) - \mathbb{E}F_m(T_n))/n$, where the associated toll function $f_m(T) = f(T)\mathbf{1}\{|T| \leq m\}$.

Generating functions

The exponential generating function $Y(x)$ of d-ary increasing trees satisfies the differential equation

$$Y'(x) = \Phi(Y(x)), \qquad Y(0) = 0,$$

where $\Phi(t) = (1+t)^d$. Explicitly, $Y(x) = (1-(d-1)x)^{-1/(d-1)} - 1$.

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where $\Phi(t) = (1+t)^d$. Explicitly, $Y(x) = (1-(d-1)x)^{-1/(d-1)} - 1$. Consider the multivariate generating function

$$Y(x,a,b) = \sum_T \frac{x^{|T|}}{|T|!} e^{aF(T)-bf(T)}.$$

We also have the differential equation

$$\frac{\partial}{\partial x} Y(x,a,a) = \sum_T \frac{x^{|T|-1}}{(|T|-1)!} e^{a(F(T)-f(T))} = \Phi(Y(x,a,0))$$

with $Y(0,a,b) = 0$.

Generating functions

We set

$$Z(x,a,b) = 1 + Y(xe^{-a\mu},a,b) = 1 + \sum_T \frac{x^{|T|}}{|T|!} e^{aF(T) - a\mu|T| - bf(T)}$$

where μ will be determined later. In particular, we have

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Note that the moment generating function of $F(T_n) - \mu n$ is

$$\mathbb{E}(\exp(aF(T_n) - a\mu n)) = \frac{[x^n]Z(x,a,0)}{[x^n]Z(x,0,0)}.$$

Moments

We represent partitions of an integer s as sequences $\ell = (\ell_1, \ell_2, \ldots)$, where ℓ_j denotes the multiplicity of j. We write $|l| = l_1 + l_2 + \cdots$ for the number of parts in the partition l.

Lemma

The function $Z^{(r)}(x,0,0)$ *satisfies the differential equation*

$$\frac{\partial}{\partial x}\Big(Z(x,0,0)^{-d} Z^{(r)}(x,0,0)\Big) = -Z(x,0,0)^{-d} H_r(x) + \sum_{s=0}^{r} \binom{r}{s} (-\mu)^{r-s} s! \sum_{\substack{\ell \in \mathcal{P}(s) \\ \ell_r \neq 1}} \frac{d!}{(d-|\ell|)!} \prod_{j \geq 1} \frac{1}{\ell_j! j!^{\ell_j}} \left(\frac{Z^{(j)}(x,0,0)}{Z(x,0,0)}\right)^{\ell_j},$$

where

$$H_r(x) = \sum_{s=1}^{r} \binom{r}{s} \sum_{T} \frac{x^{|T|-1}}{(|T|-1)!} (F(T) - \mu|T|)^{r-s} (-f(T))^s.$$

Finite support

Theorem

If the toll function f has finite support, i.e. there exists a constant K such that $f(T) = 0$ whenever $|T| > K$, then the centred moments of the functional F are asymptotically given by

$$\mathbb{E}((F(T_n) - \mu n)^r) = \begin{cases} (r-1)!!\sigma^r n^{r/2} + O\left(n^{r/2-1}\right) & r \text{ even,} \\ O\left(n^{(r-1)/2}\right) & r \text{ odd.} \end{cases}$$

Consequently, if $\sigma \neq 0$, then the renormalised random variable

$$\frac{F(T_n) - \mu n}{\sqrt{\sigma^2 n}}$$

converges weakly to a standard normal distribution.

Sketch of the proof

If the toll function f has finite support then

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is a polynomial in x for every $r \geq 1$. From the differential equation of Z, we have

$$Z^{(1)}(x,0,0) = Z(x,0,0)^d \int_0^x \Big(- Z(w,0,0)^{-d} H_1(w) - \mu \Big)\, dw.$$

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Now, we choose the constant μ as follows:

$$\mu = -(d-1) \int_0^{1/(d-1)} Z(w,0,0)^{-d} H_1(w)\, dw.$$

Sketch of the proof

We deduce that

$$Z^{(1)}(x,0,0) = \frac{\mu}{d-1}(1-(d-1)x)^{-1/(d-1)} + \mathcal{O}\Big(|1-(d-1)x|^{-d/(d-1)}\Big).$$

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Moreover, we can prove by induction that for each positive integer p there exists a constant c_p such that

$$Z^{(2p)}(x,0,0) = c_p(1-(d-1)x)^{-1/(d-1)-p} + \mathcal{O}\Big(|1-(d-1)x|^{-1/(d-1)-p+1}\Big)$$

and

$$Z^{(2p+1)}(x,0,0) = \mathcal{O}\Big(|1-(d-1)x|^{-1/(d-1)-p}\Big).$$

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The estimates of the centred moments are obtained by singularity analysis.

General: Mean and variance

For general toll functions, note that only the trees of order $\leq n$ contribute to the distribution of $F(T_n)$, so we can assume without loss of generality that $f(T) = 0$ whenever $|T| > n$.

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$$\mathbb{E}F(T_n) = (d(d-1)n + d) \sum_{m<n} \frac{\mathbb{E}f(T_m)}{((d-1)m+1)((d-1)m+d)} + \mathbb{E}F(T_n).$$

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If the toll function f satisfies (C1), (C2), and (C3), then the mean satisfies the estimate

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The same argument applies for the variance.

Applications

Given a fixed tree S, let $F(T)$ be the number of occurrences of the tree S on the fringe of T. The corresponding toll function is

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Our theorem applies for the random variable $F(T_n)$, where T_n is a uniform random d-ary increasing tree of order n, with

$$\mu = \frac{d-1}{\prod_{j=1}^{|S|}((d-1)j+d)}$$

$$\sigma^2 = -\mu^2\left(2|S| + \frac{1}{d-1}\right) + \mu + \frac{d(d-1)^{1-2|S|}}{(|S|-1)!^2}\int_0^1 \phi_{|S|}(x)^2\,dx.$$

Applications

Given a fixed positive k, let $F(T)$ be the number of fringe subtrees of order k of the tree T. The corresponding toll function is

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$$\sigma^2 = -\mu^2\left(2k + \frac{1}{d-1}\right) + \mu + \frac{d(d-1)^{1-2k}Y_k^2}{(k-1)!^2}\int_0^1 \phi_k(x)^2\,dx.$$

This result was already obtained by Fuchs (2012).

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Therefore, $F(T) = \log(1 + s(T))$ is an additive functional, since

$$\log(1 + s(T)) = \sum_j \log(1 + s(B_j)) + \log(1 + s(T)^{-1}).$$

The toll function $f(T) = \log(1 + s(T)^{-1})$ satisfies all our conditions, so our theorem applies in this case as well.

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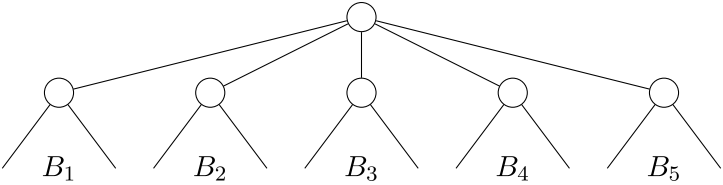
Wagner (2015) considered this functional as an auxiliary quantity for studying the number of subtrees of a tree.

Applications

Let $F(T)$ be the logarithm of the size of the automorphism group of T. The corresponding toll function is $f(T) = \log R(T)$, where $R(T)$ is the group of symmetries of the collection of the branches B_1, B_2, B_3, ...



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Therefore, for $d \geq 3$, the toll functional satisfies (C2) and (C3) since

$$\mathbb{E} f(T_n) = \mathcal{O}(n^{-2/(d-1)} \log n).$$

Our theorem applies.

GPORTs

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Theorem

Let T_n be a random GPORT (with fixed parameter α) with n vertices. If the toll function $f(T)$ satisfies (C1), (C2) and (C3), then there exist constants μ and σ such that the mean and variance of $F(T_n)$ are asymptotically

$$\mathbb{E}(F(T_n)) = \mu n - \frac{\mu}{\alpha+1} + o(1), \quad \mathrm{Var}(F(T_n)) = \sigma^2 n + o(n).$$

If $\sigma \neq 0$, then the renormalised random variable $(F(T_n) - \mu n)/\sqrt{\sigma^2 n}$ converges weakly to a standard normal distribution.

Thanks

THANK YOU.