q-Quasiadditive Functions

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Optimal $\{0, 1, -1\}$-Representations

- Base 2 representations
- With digits 0, 1 and -1

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- Cryptography on elliptic curves: Scalar multiplication
- Always using the same expansion $\rightsquigarrow$ potential attack

Optimal $\{0, 1, -1\}$-Representations

Lemma (Grabner–Heuberger 2006)

$$r_{\mathrm{OPT}}(0) = u_2(0) = \cdots = u_5(0) = 1,$$
$$r_{\mathrm{OPT}}(1) = u_2(1) = 1, \; u_3(1) = u_4(1) = u_5(1) = 0,$$

and

$$\begin{aligned}
r_{\mathrm{OPT}}(2n) &= r_{\mathrm{OPT}}(n), & r_{\mathrm{OPT}}(2n+1) &= u_2(n) + u_4(n+1),\\
u_2(2n) &= r_{\mathrm{OPT}}(n), & u_2(2n+1) &= u_3(n),\\
u_3(2n) &= u_2(n), & u_3(2n+1) &= 0,\\
u_4(2n) &= r_{\mathrm{OPT}}(n), & u_4(2n+1) &= u_5(n+1),\\
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\end{aligned}$$

$$r_{\mathrm{OPT}}(n) = \langle 1, 1, 1, 2, 1, 1, 2, 1, 1, 1, 1, 3, 2, 3, 1, 1, \ldots \rangle$$

Optimal $\{0, 1, -1\}$-Representations

Recursion implies

$$r_{\mathrm{OPT}}(2^3 n) = r_{\mathrm{OPT}}(n),$$
$$r_{\mathrm{OPT}}(2^3 n + 1) = r_{\mathrm{OPT}}(n) r_{\mathrm{OPT}}(1) = r_{\mathrm{OPT}}(n).$$

Lemma (K.–Wagner 2016)

More generally for $0 \le b < 2^\ell$, $a, \ell \ge 0$*:*

$$r_{\mathrm{OPT}}(2^{\ell+3} a + b) = r_{\mathrm{OPT}}(a) r_{\mathrm{OPT}}(b).$$

Cellular Automata

- Cubic lattice: each cell is either
  ON = black or OFF = white
- From generation n to $n+1$: a cells turns on or off depending on its neighbourhood at generation n according to some rules

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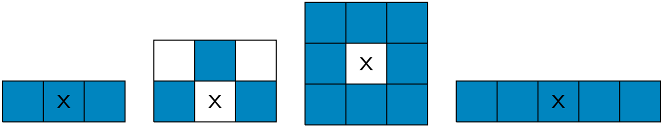
- From generation n to $n+1$: a cells turns on or off depending on its neighbourhood at generation n according to some rules
- Odd-rule cellular automata:
  - Define a neighbourhood
  - If an odd number of neighbours was ON in generation n, then the cell is ON in generation $n+1$.

Examples of neighbourhoods:



Cellular Automata

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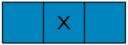
Neighbourhood: 

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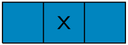
Neighbourhood: 

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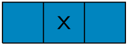
Neighbourhood: 

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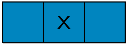
Neighbourhood: 

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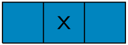
Neighbourhood: 

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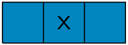
Neighbourhood: 

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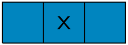
Neighbourhood: 

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- $a(n) =$ number of on-cells in generation n

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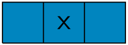
Neighbourhood: 

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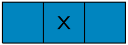
Neighbourhood: 

$$a(n) = \langle 1, 3, \qquad\qquad \ldots \rangle$$

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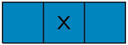
Neighbourhood: 

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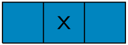
Neighbourhood: 

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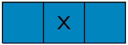
Neighbourhood: 

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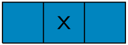
Neighbourhood: 

$$a(n) = \langle 1, 3, 3, 5, 3, 9, \ldots \rangle$$

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Run Length Transform

Definition

$\mathcal{L}(n)$ is the list of lengths of maximal runs of 1s in the binary expansion of n.

The run length transform of a given sequence $s_1, s_2, \ldots$ is

$$t(n) = \prod_{i \in \mathcal{L}(n)} s_i.$$

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$$1910 = (11101110110)_2$$

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Implies for $0 \leq m < 2^\ell$

$$t(2^{\ell+1} n + m) = t(n) t(m).$$

Run Length Transform and Cellular Automata

Theorem (Sloane 2015)

For all 3×3 neighbourhoods, $a(n)$ is the run length transform of its subsequence

$$\langle a(0), a(1), a(3), a(7), a(15), \ldots, a(2^n - 1), \ldots \rangle.$$

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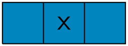
Neighbourhood: 

Subsequence = Jacobsthal numbers:

$$s_i = \langle 1, 3, 5, \ldots, (2^{n+2} - (-1)^n)/3, \ldots \rangle$$
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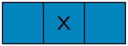
Neighbourhood: 

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$$9 = a(5) = a((101)_2) = s_1^2 = 3^2$$

Common Property: 2-Quasimultiplicative

Definition

Let $r \geq 0$.
$f(a)$ is called 2-quasimultiplicative if

$$f(2^{\ell+r}a + b) = f(a)f(b)$$

holds for all ℓ, $a, b \geq 0$ and $0 \leq b < 2^{\ell}$.

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- Also 2-quasiadditive (connected via logarithm):

$$f(2^{\ell+r}a + b) = f(a) + f(b)$$

Examples

Example (Optimal $\{0, 1, -1\}$-Representations)

$$r_{\mathrm{OPT}}(2^{\ell+3}a + b) = r_{\mathrm{OPT}}(a) r_{\mathrm{OPT}}(b).$$

is 2-quasimultiplicative with parameter $r = 3$.

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Example (Cellular Automaton and Run Length Transform)

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Run length transform

$\Leftrightarrow$

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Distribution of Strongly 2-Multiplicative Functions

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- Bellman–Shapiro 1948, Gel'fond 1967, Delange 1972, Cateland 1992, Bassily–Kátai 1995, Drmota 2000, ...

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- We want something similar for 2-quasimultiplicative functions.

Elementary Property

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Theorem (K.–Wagner 2016)

2-*quasimultiplicative with parameter* $r = 3$*:*

$$204\,280\,974 = (1100001011010001010010001110)_2$$

corresponds to

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$$204\,280\,974 = (1100001011010001010010001110)_2$$

corresponds to

$$3, \quad 45, \quad 41 \quad \textit{and} \quad 7$$

$$f(204\,280\,974) = f(3)f(45)f(41)f(7)$$

Notation

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- Symbolic method leads to

$$F(x,t) = \frac{1+(1+x+\cdots+x^{r-1})B(x,t)}{1-x-x^r B(x,t)}$$

Central Limit Theorem

Under certain growth conditions on f, the dominant root of $1 - x - x^r B(x,t)$ behaves nicely.

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Singularity analysis and Quasi-Power theorem leads to:

Theorem (K.–Wagner 2016)

N_k = random integer in $\{0, 1, \ldots, 2^k - 1\}$
Under certain growth conditions on f and $f > 0$

$$\mathbb{E}(\log f(N_k)) = \frac{B_t(1/2, 0)}{2^{2r}} k + O(1)$$

and linear variance $\mathbb{V}(\log f(N_k)) = \sigma^2 k + O(1)$
If f is not constant, then $\sigma^2 \neq 0$ and

$$\frac{\log f(N_k) - \mu k}{\sigma \sqrt{k}} \xrightarrow{d} \mathcal{N}(0,1)$$

Optimal $\{0, 1, -1\}$-Representations

$$r_{\mathrm{OPT}}(n) = \langle 1, 1, 1, 2, 1, 1, 2, 1,\\ 1, 1, 1, 3, 2, 3, 1, 1, \ldots \rangle$$

- $\log r_{\mathrm{OPT}}(N_k)$ is asymptotically normally distributed
- $\mu \approx 0.060829$, $\sigma^2 \approx 0.038212$

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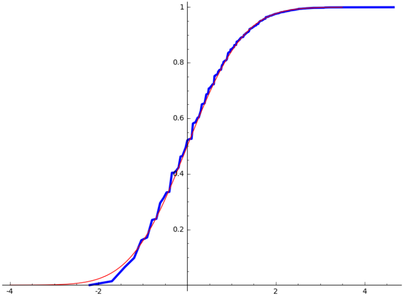

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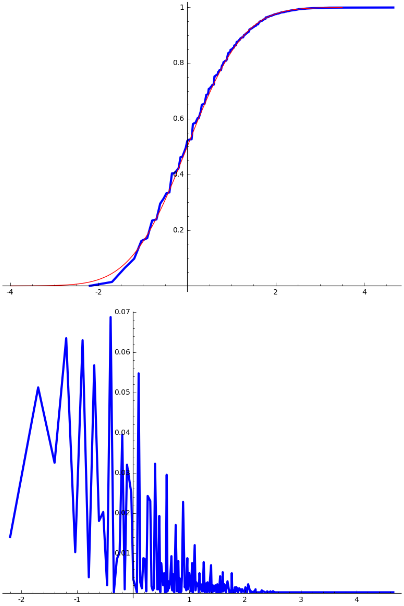
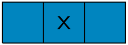


Cellular Automaton

- Neighbourhood: 

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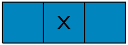
- Neighbourhood: 

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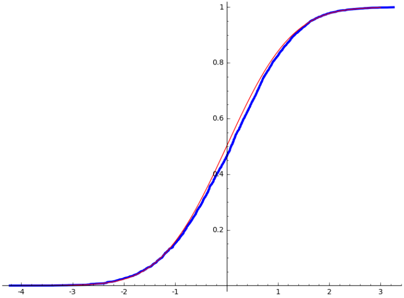

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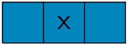
- Neighbourhood: 

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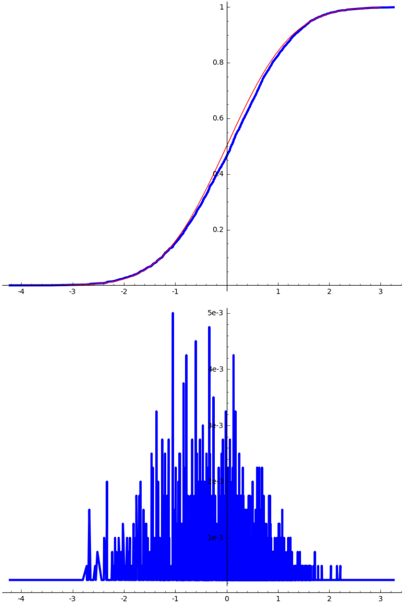

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r_{\mathrm{OPT}}(2n) &= r_{\mathrm{OPT}}(n), & r_{\mathrm{OPT}}(2n+1) &= u_2(n) + u_4(n+1),\\
u_2(2n) &= r_{\mathrm{OPT}}(n), & u_2(2n+1) &= u_3(n),\\
u_3(2n) &= u_2(n), & u_3(2n+1) &= 0,\\
u_4(2n) &= r_{\mathrm{OPT}}(n), & u_4(2n+1) &= u_5(n+1),\\
u_5(2n) &= u_4(n), & u_5(2n+1) &= 0
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Definition

2-regular sequences are defined by

$$f((n_L \cdots n_0)_2) = \mathbf{u}^t \prod_{i=0}^{L} M_{n_i} \mathbf{v}$$

Optimal $\{0, 1, -1\}$-Representations

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$$f(13) = f((1101)_2) = \mathbf{u}^t M_1 M_0 M_1 M_1 \mathbf{v}$$

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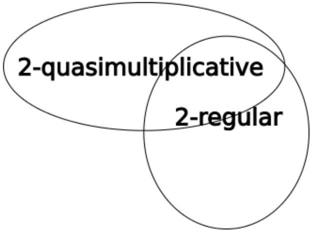
$$

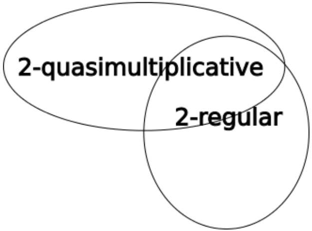
$$
M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\quad
M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}
$$

$\mathbf{u}^t = (1, 0, 0, 0, 0, 0)$ and $\mathbf{v} = (1, 1, 1, 1, 0, 0)^t$

2-Regular = 2-Quasimultiplicative?



2-Regular $=$ 2-Quasimultiplicative?

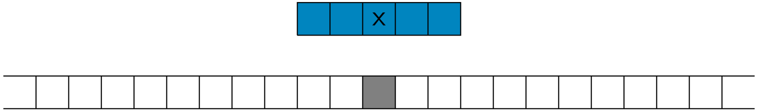


Theorem (K.–Wagner 2016)

f is 2-regular with minimal matrices and leading zeros to not change anything.

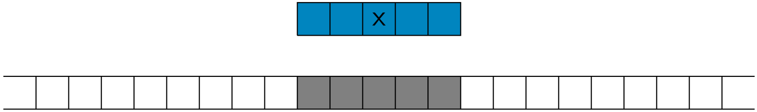
$$f \text{ is 2-quasimultiplicative with parameter } r.$$
$$\Leftrightarrow$$
$$M_0^r = \mathbf{v}\mathbf{u}^t$$

Larger Neighbourhoods and Cellular Automata



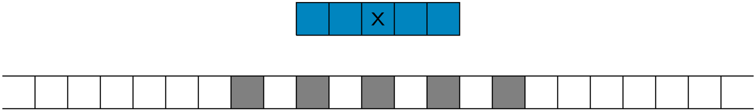
$$a(n) = \langle 1, \qquad \ldots \rangle$$

Larger Neighbourhoods and Cellular Automata



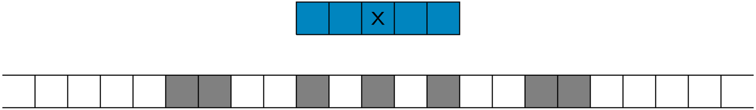
$$a(n) = \langle 1, 5, \qquad \ldots \rangle$$

Larger Neighbourhoods and Cellular Automata



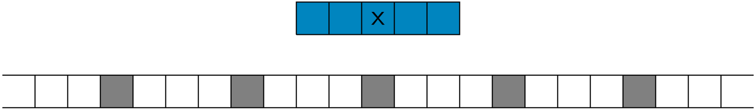
$$a(n) = \langle 1, 5, 5, \quad \dots \rangle$$

Larger Neighbourhoods and Cellular Automata



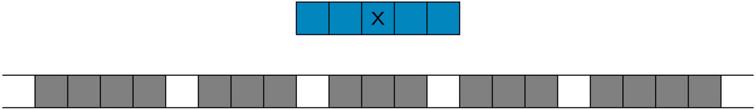
$$a(n) = \langle 1, 5, 5, {\color{blue}7}, \qquad \ldots \rangle$$

Larger Neighbourhoods and Cellular Automata



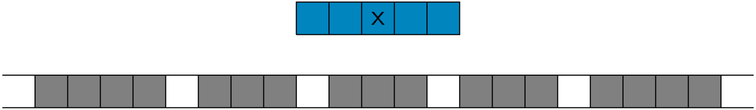
$$a(n) = \langle 1, 5, 5, 7, 5, \quad \ldots \rangle$$

Larger Neighbourhoods and Cellular Automata



$$a(n) = \langle 1, 5, 5, 7, 5, 17, \ldots \rangle$$

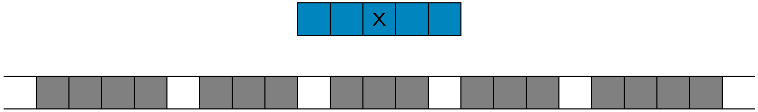
Larger Neighbourhoods and Cellular Automata



$$a(n) = \langle 1, 5, 5, 7, 5, 17, \ldots \rangle$$

$$2485 = a(7473) = a((1110100110001)_2)$$

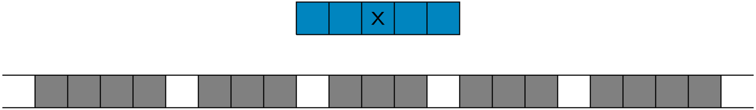
Larger Neighbourhoods and Cellular Automata



$$a(n) = \langle 1, 5, 5, 7, 5, 17, \ldots \rangle$$

$$2485 = a(7473) = a((1110100110001)_2) = a(29)a(3)a(1) = 71 \cdot 7 \cdot 5$$

Larger Neighbourhoods and Cellular Automata



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$\Rightarrow$ 2-quasimultiplicative with parameter $r = 2$

Conclusion

- Expected Value and Variance of 2-quasimultiplicative and 2-quasiadditive functions
- Central limit theorem
- Connections to 2-regular sequences
- Applications for analysing cellular automaton and the number of optimal representations

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Thank you for your attention!