Limit laws of vertex degree distribution in planar maps

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Outline

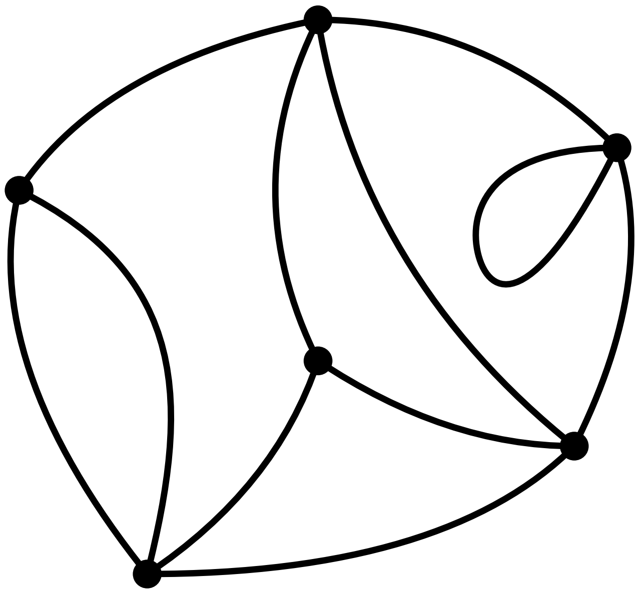
0. Introduction
1. Main result
2. A degree-preserving bijection
3. Asymptotic expansion
4. Proof of the main theorem
5. And beyond!

Introduction

Planar map =

Graph embedded on the sphere

($\simeq$ drawn on the plane without crossings)



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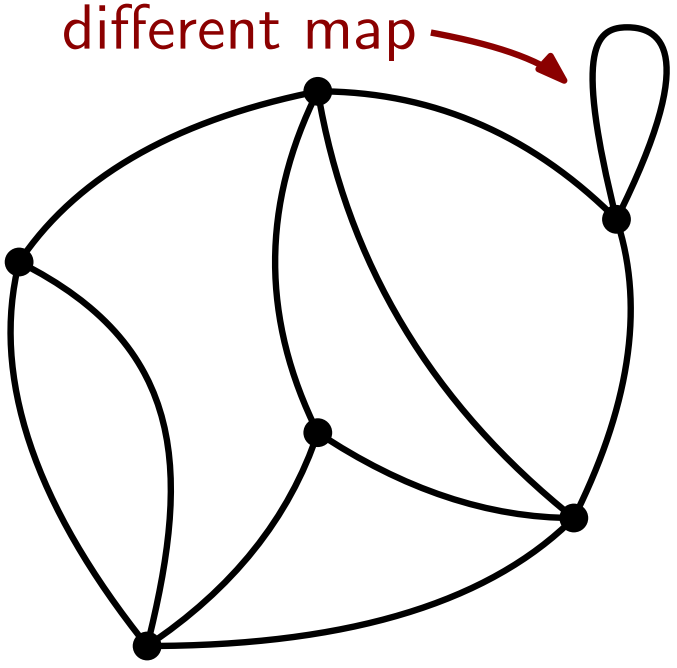

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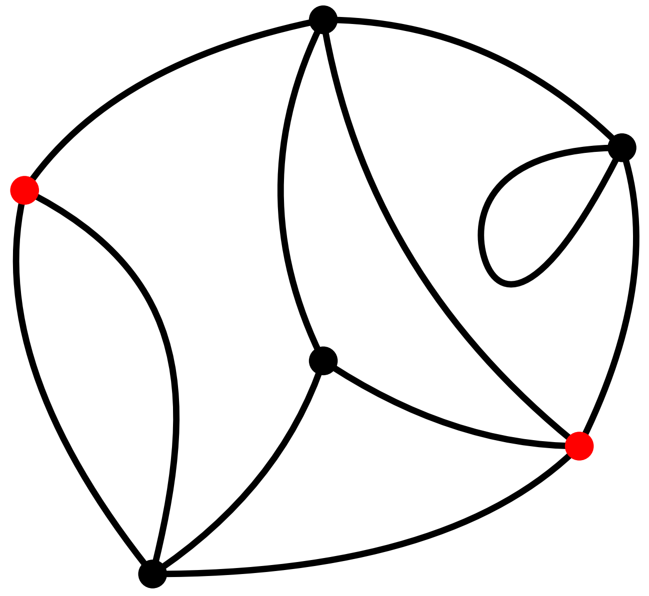

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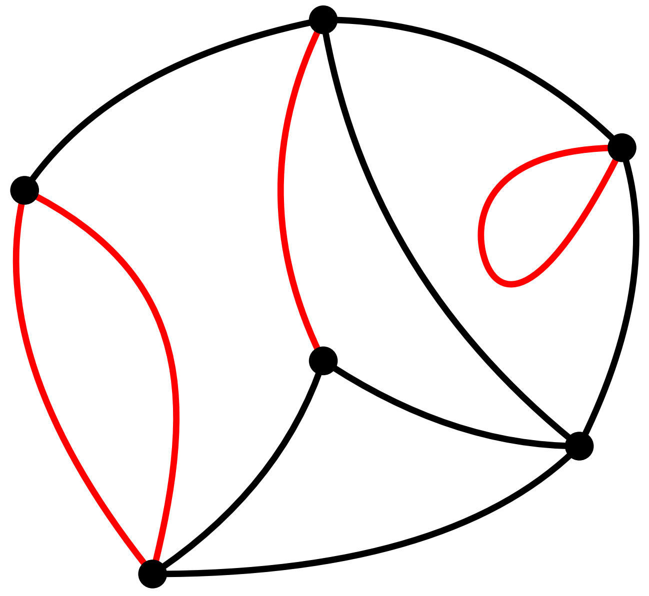
vertices, edges (multiple, loops. . .)



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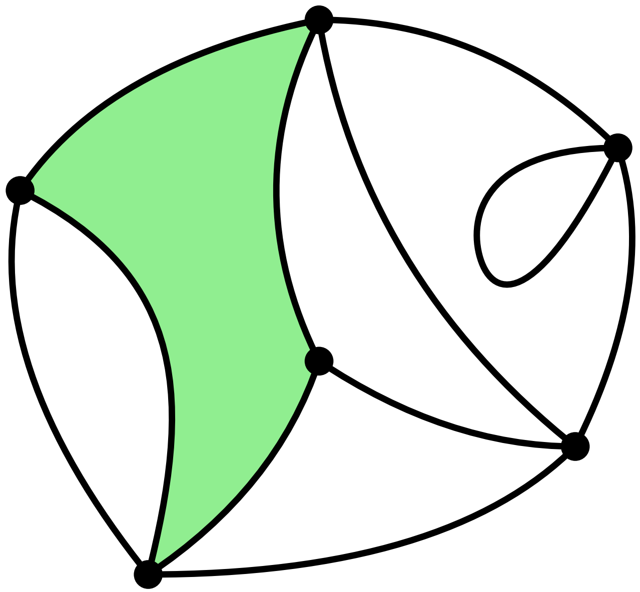

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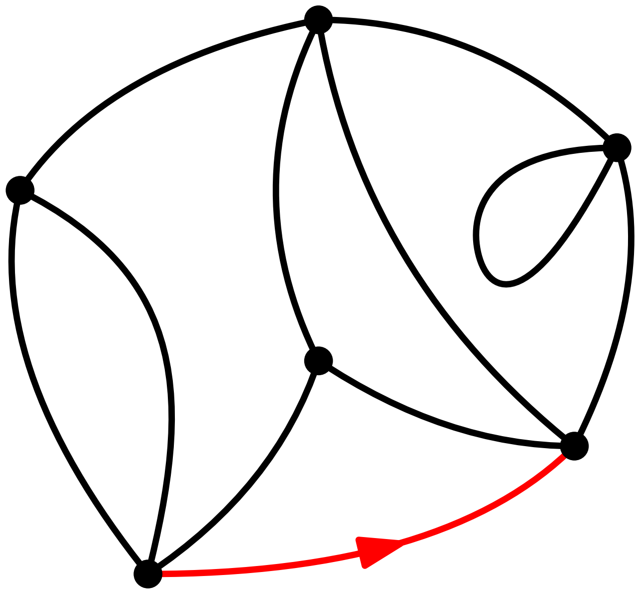

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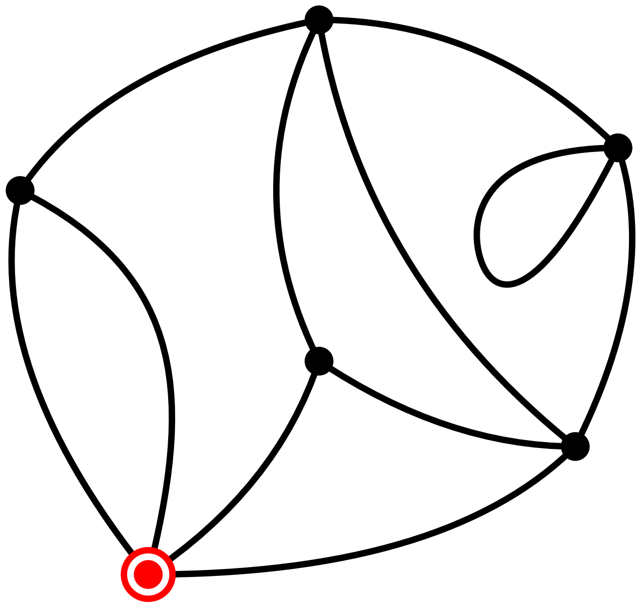

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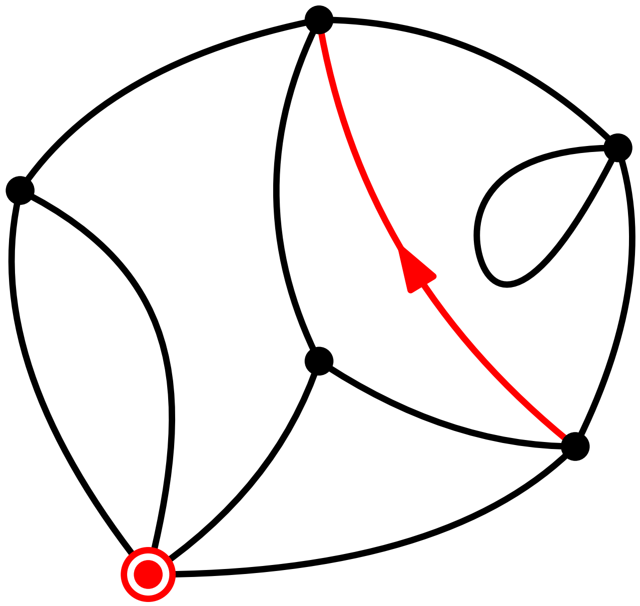

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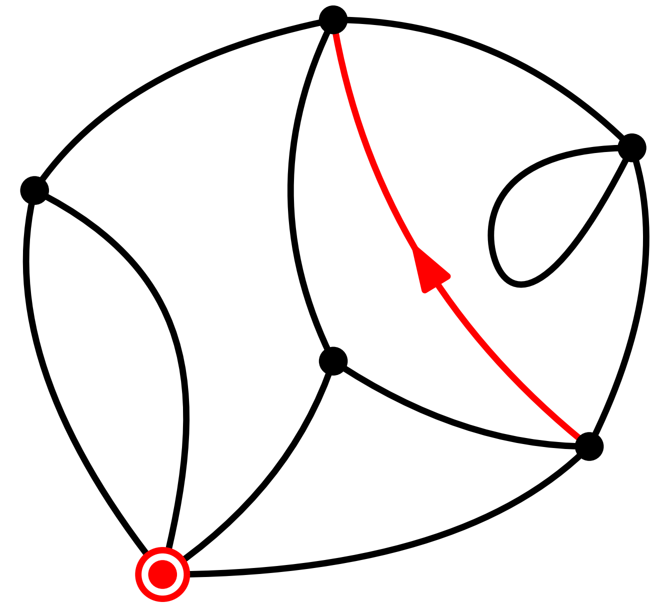

$\mathcal{M}$ family of rooted planar maps

D, set of positive integers, finite or infinite

$\mathcal{M}_D \subset \mathcal{M}$ where the vertex degrees are restricted to D

ex: $D = \{3\} \to$ cubic maps ($\simeq$ triangulations)

$D = \{2n, n \geq 1\} \to$ Eulerian maps ($\simeq$ bipartite maps)

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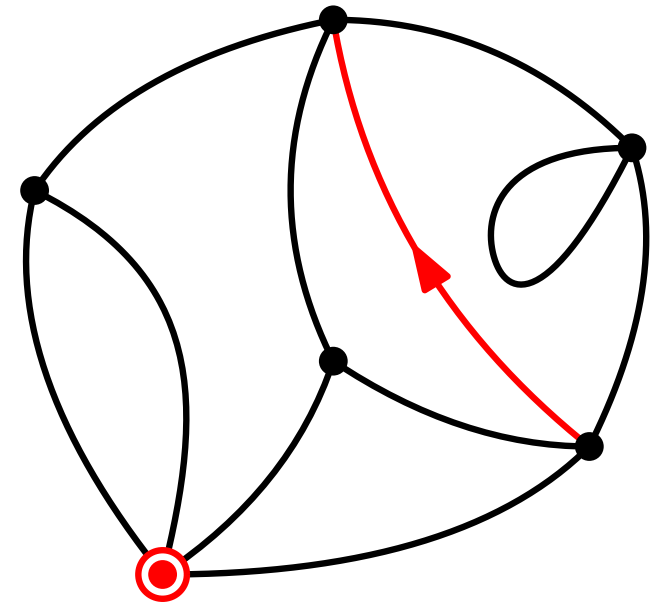

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Asymptotic behaviour of $\mathcal{M}_D$?

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M_n number of maps with n edges

Tutte (60s): $M_n = \frac{2\cdot 3^n}{n+2}\mathrm{Cat}(n) \Rightarrow M_n \sim \frac{2}{\sqrt{\pi}} n^{-5/2} 12^n$

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One-dimensional Central Limit Theorem for general maps

[Drmota, Panagiotou'12]

Multi-dimensional CLT for bipartite maps

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One-dimensional Central Limit Theorem for general maps
[Drmota, Panagiotou'12]

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Let's go more general!

Main result

Theorem [C.-Drmota-Klausner'16]

$\forall D$ finite or infinite, $D \not\subseteq \{1,2\}$
$M_{D,n}$ number of maps in $\mathcal{M}_D$ with n edges
$d = \gcd\{i : 2i \in D\}$ if D even, $d = 1$ otherwise
$\mathbf{X}_n = (X_n^{(d)})_{d\in D} (n \equiv 0 \bmod d)$

Then there exist positive constants c_D, ρ_D with:

$$M_{D,n} \sim c_D n^{-5/2} \rho_D^{-n}, \quad n \equiv 0 \bmod d$$

Furthermore, there exists a positive constant μ_D such that:

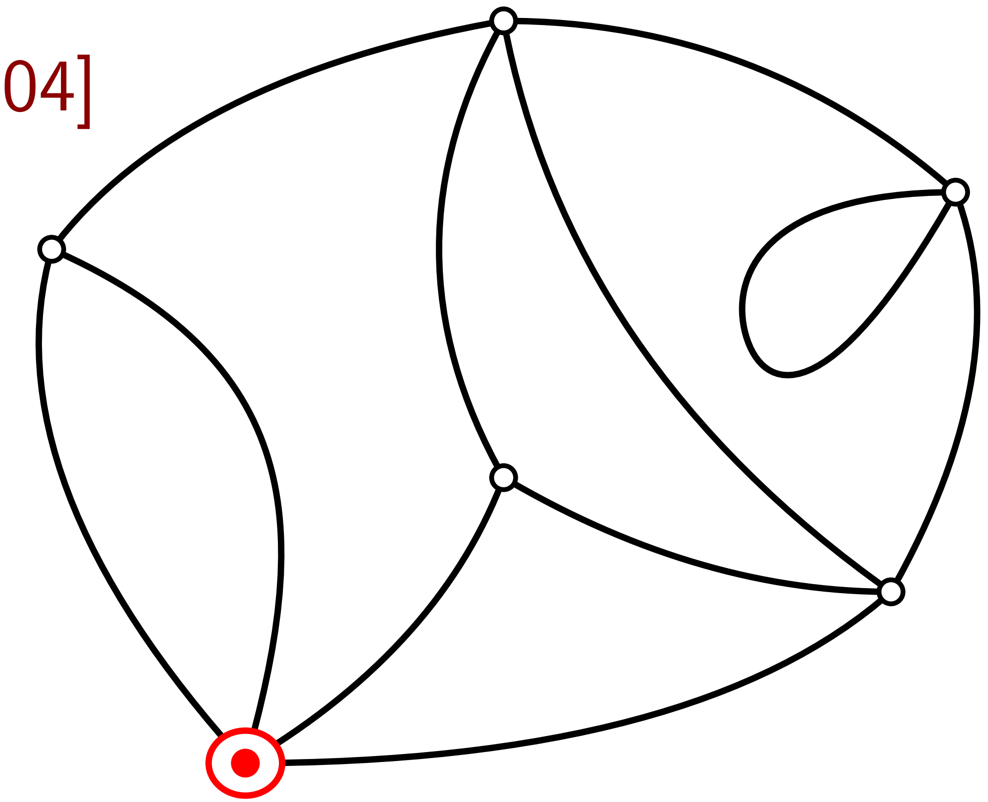
$$E[X_n^{(d)}] \sim \mu_D n$$

and $\frac{1}{\sqrt{n}}(\mathbf{X}_n - E(\mathbf{X}_n)), \quad n \equiv 0 \bmod d$, converges weakly to a centered Gaussian random variable $\mathbf{Z}$ (in ℓ^2).

A degree-preserving bijection

[Bouttier-Di Francesco-Guitter'04]

1. Pointed map



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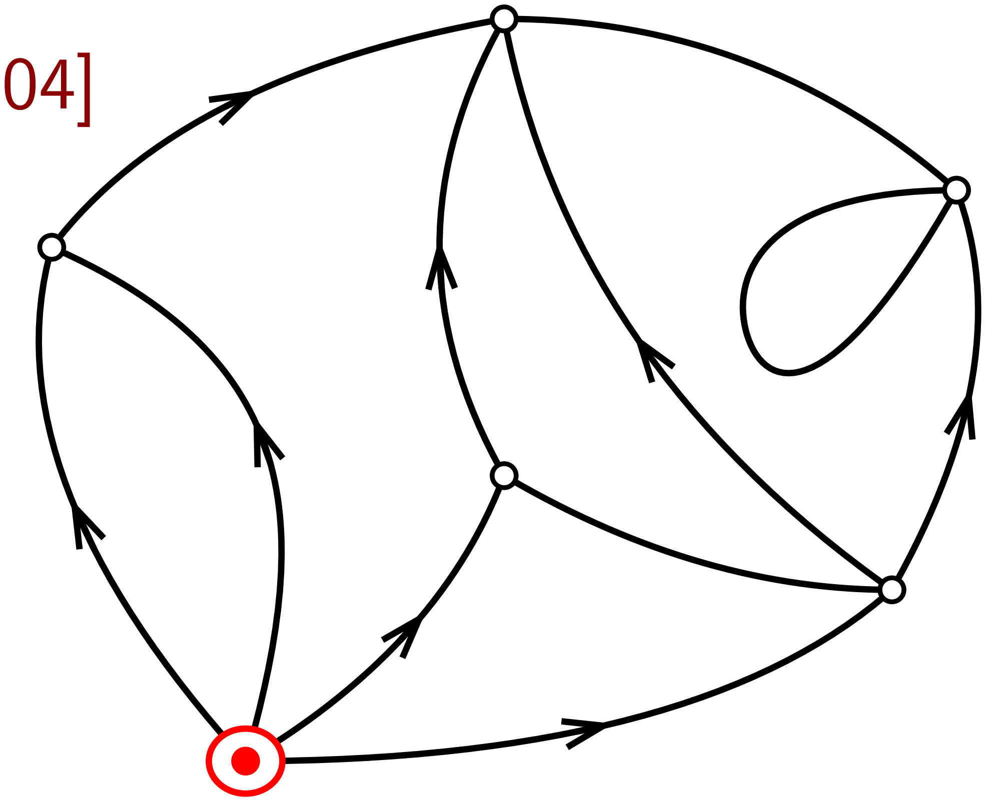

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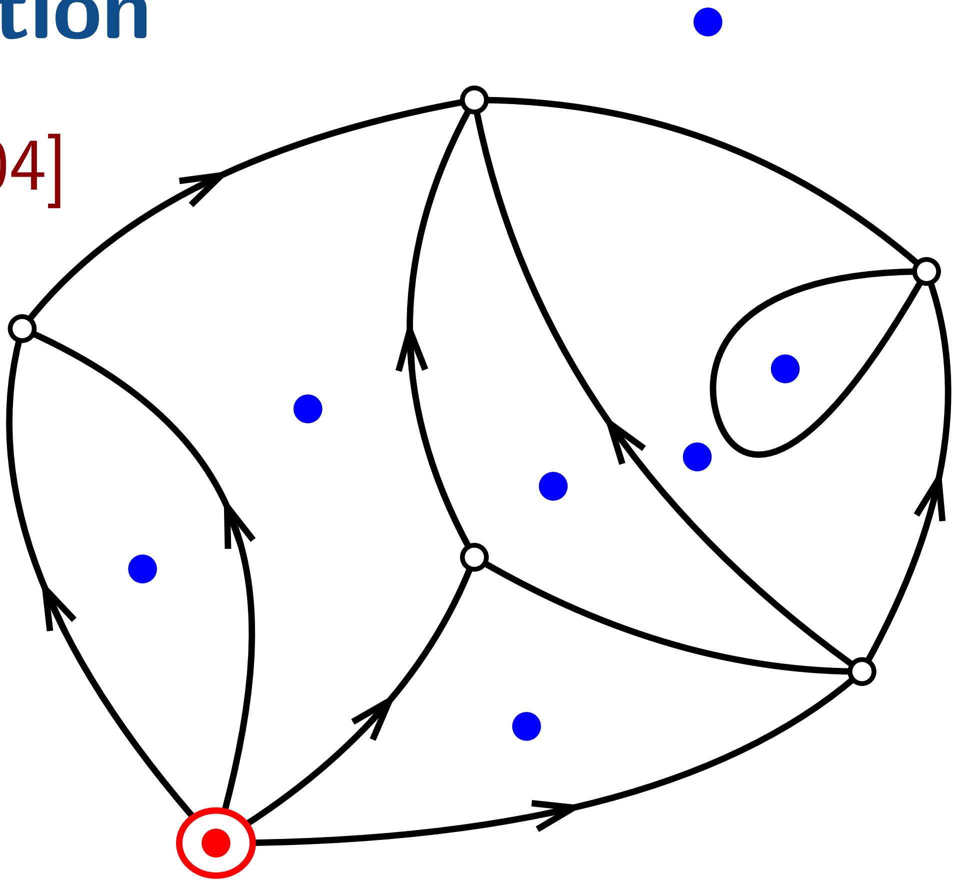

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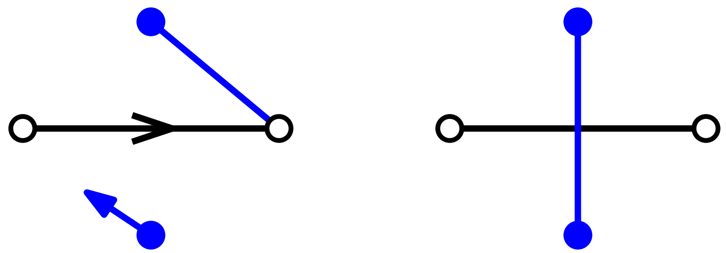

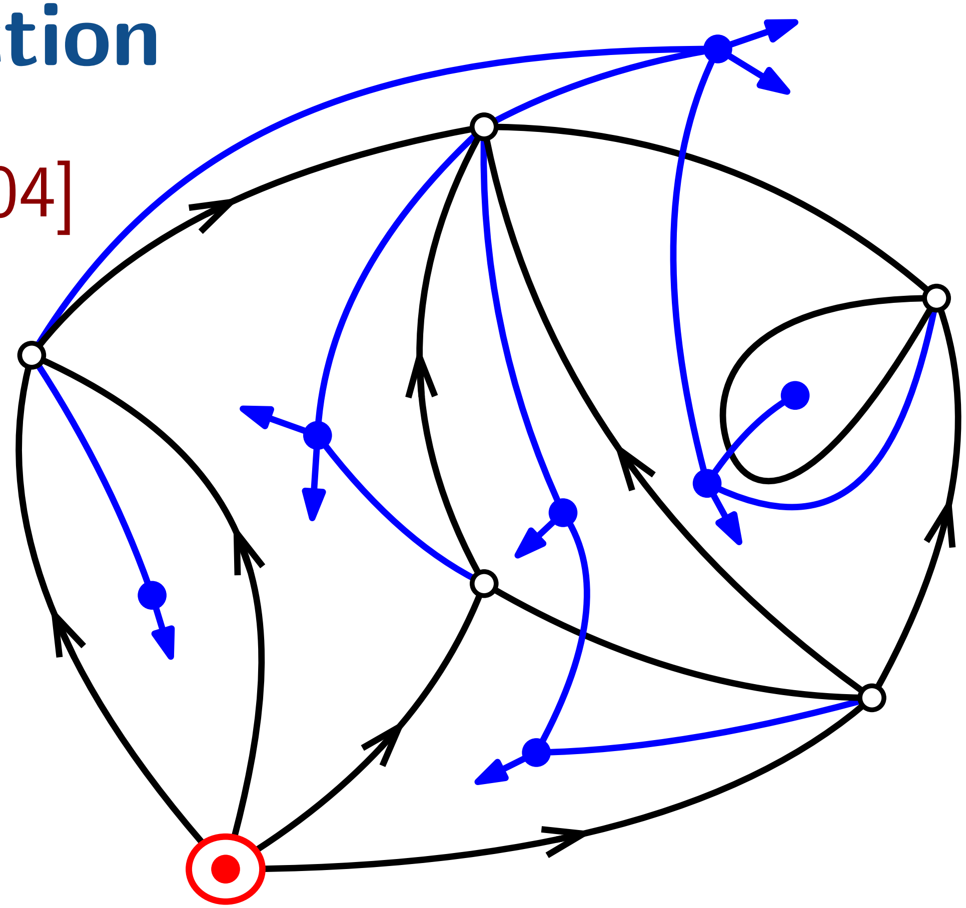

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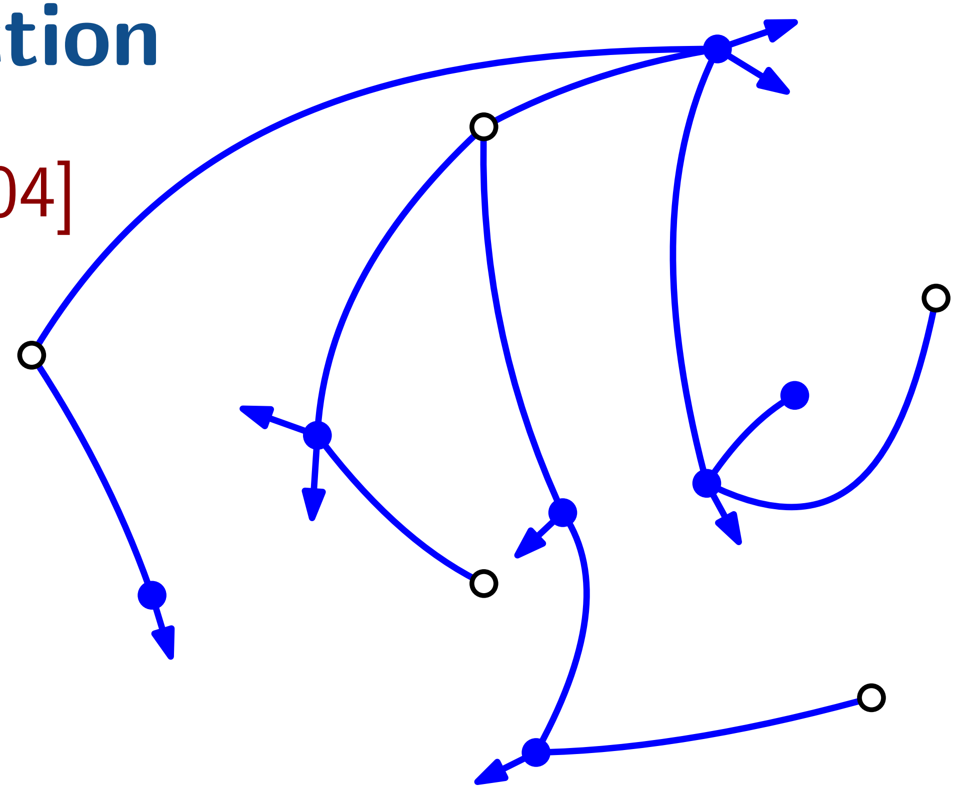

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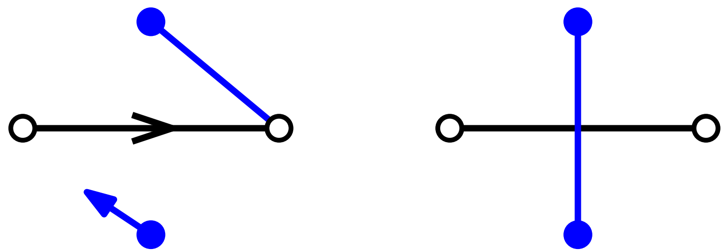


5. Erase map edges and pointed vertex

Bijection between **pointed maps** and **mobiles**

edges $\leftrightarrow$ edges

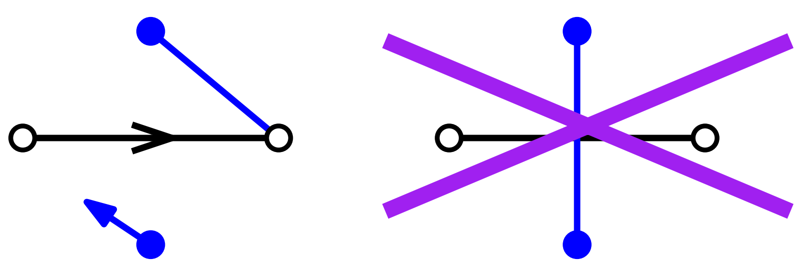
non-pointed vertices $\leftrightarrow$ white vertices

face of degree d $\leftrightarrow$ blue vertices of degree d

A degree-preserving bijection

[Bouttier-Di Francesco-Guitter'04]

1. Pointed **bipartite** map
2. Geodesic orientation of edges
3. New blue vertex in each face
4. For each edge, apply:



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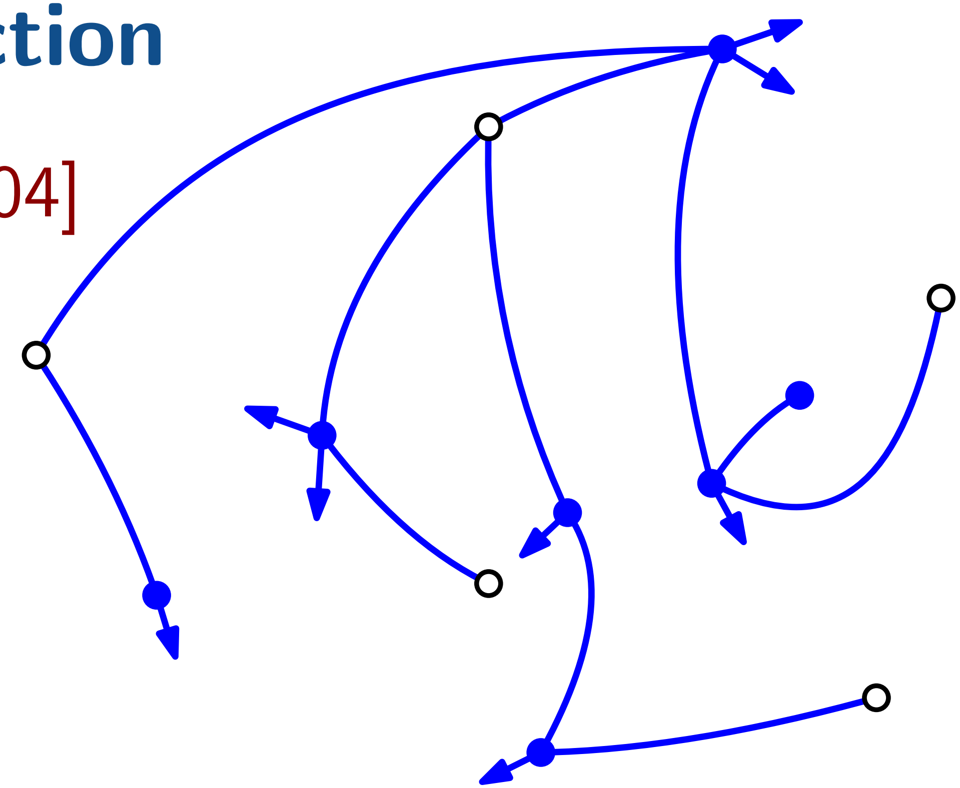

Bijection between **pointed maps** and **bipartite mobiles**

(only white-black edges)

edges $\leftrightarrow$ edges

non-pointed vertices $\leftrightarrow$ white vertices

face of degree d $\leftrightarrow$ blue vertices of degree d

A degree-preserving bijection

[Bouttier-Di Francesco-Guitter'04]

$R = R(t, z, (x_{2i})_{i\geq 1})$ generating series of bipartite mobiles

$M = M(t, z, (x_{2i})_{i\geq 1})$ generating series of bipartite maps

t: vertices/white vertices, z: edges, x_{2i}: faces/blue vertices of degree $2i$

$$R = tz + z \sum_{i\geq 1} x_{2i} \binom{2i-1}{i} R^i \qquad \frac{\partial M}{\partial t} = 2\,(R/z - t)$$

$$R_D(z,t) = tz + z \sum_{2i\in D} \binom{2i-1}{i} R_D^i \qquad \frac{\partial M_D}{\partial t} = 2\,(R_D/z - t)$$

$R_D(z,t) = F(t, z, R_D)$ where F is a formal power series with nonnegative integer coefficients

Asymptotic expansion

$$R_D(z,t) = F(z,t,R_D) = tz + z \sum_{2i \in D} \binom{2i-1}{i} R_D^i \qquad (*)$$

Lemma:

$\exists \rho(t)$ analytic near $t=1$, with $\rho(1) > 0$ and $\rho'(1) \neq 0$,
$\exists g(z,t), h(z,t)$ analytic near $(z=\rho(1), t=1)$, with $h(\rho(1),1) > 0$,
such that the unique solution R_D of $(*)$ analytic at $(0,0)$ is expressed as:

$$R_D(z,t) = g(z,t) - h(z,t)\sqrt{1 - \frac{z}{\rho(t)}}$$

Proof: [Drmota, Random trees'09]

Show that $\begin{cases} R_0 = F(\rho, 1, R_0) \\ 1 = F_R(\rho, 1, R_0) \end{cases}$ admits positive solutions (R_0, ρ)

- D finite: F polynomial, easy
- D infinite: $\begin{cases} R_0 = F(\rho, 1, R_0) \\ 1 = F_R(\rho, 1, R_0) \end{cases} \Rightarrow H(R_0) = \sum_{2i \in D} (i-1)\binom{2i-1}{i} R_0^i = 1$

$(i-1)\binom{2i-1}{i} \sim \frac{4^i \sqrt{i}}{2\sqrt{\pi}} \Rightarrow H(x)$ has radius $1/4$, $H(x) \to_{x \to 1/4^-} \infty$

$+$ analytic conditions on partial derivatives of F at $(\rho, 1, R_0)$

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Lemma:

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such that the unique solution R_D of $(*)$ analytic at $(0,0)$ is expressed as:

$$M_D(z,t) = g_2(z,t) + h_2(z,t)\left(1 - \frac{z}{\rho(t)}\right)^{3/2}$$

$$\Rightarrow \exists c_D > 0, M_{D,n} = [z^n]M_D(z,1) \sim c_D n^{-5/2} \rho^{-n}, \quad n \equiv 0 \bmod d$$

Transfer lemma

Central limit theorem

$$R_D(z,t,(x_{2i})) = F(z,t,(x_{2i}),R_D) = tz + z\sum_{2i\in D} x_{2i}\binom{2i-1}{i}R_D^i \qquad (*)$$

- D finite: Apply [Drmota, Random trees, Theorem 2.25]:

  $$R_D(z,t,(x_{2i})) = g(z,t,(x_{2i})) - h(z,t,(x_{2i}))\sqrt{1-\frac{z}{\rho(t,(x_{2i}))}}$$

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  implies multivariate CLT for random vector $\mathbf{X}_n$

- D infinite: Apply [Drmota-Gittenberger-Morgenbesser, Theorem 3]:

  similar expansion

  $+$ condition $X_n^{(i)} = 0$ for $i > cn$ (for some $c > 0$)

  $+$ tightness of $\mathbf{X}_n \Rightarrow$ moment conditions

  ↳ compute derivatives of F at $(\rho, 1, (1)_{2i\in D}, R_0)$

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  → follows directly from $\sum_{i\geq 1} i^K\binom{2i-1}{i}y^i < \infty$

Central limit theorem

What about general case?

System of two equations instead of one:

$$\begin{cases} L_D = & z\sum_{i\in D} y_i \sum_m B_{2m-i-1,m} L_D^{2m-i-1} R_D^m \\ R_D = & zt + z\sum_{i\in D} y_i \sum_m B^{(+1)}_{2m-i-2,m} L_{,D}^{2m-i-2} R_D^{m+1} \end{cases}$$

positive strongly connected,

can be reduced to a single (implicit) equation [Drmota]

$$R_D = G(z,t,(y_i)_{i\in D}, f(z,t,(y_i)_{i\in D}, R_D), R_D)$$

with $f(z,t,(x_i)_{i\in D}, r)$ analytic.

+ Re-use the same method as bipartite case.

And beyond?

Another extension: higher genus

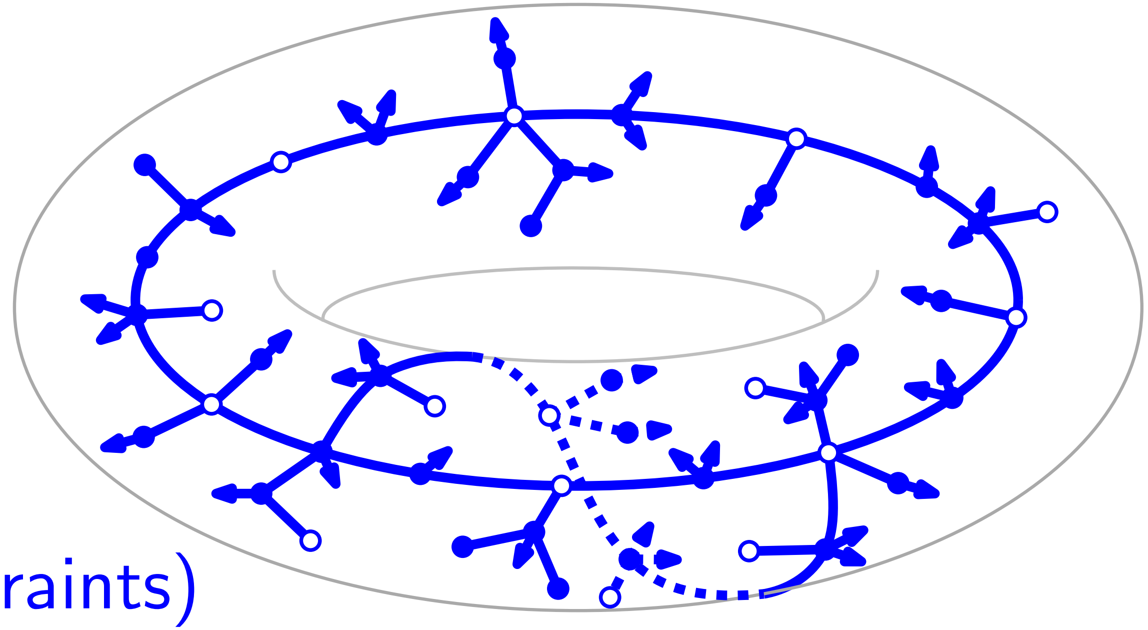
Maps on a surface of genus $g > 0$, ex. Torus

BDG bijection still works (on orientable surfaces)!

g-mobiles
$\simeq$ one-faced map

similar to planar mobiles

but different (global constraints)



Enumeration *via* scheme decomposition
[Chapuy-Marcus-Schaeffer'08]

Much more involved generating series

Still needs to study the asymptotics (work in progress)