Solutions of First Order Linear Partial Differential Equations Related to Urn Models and Central Limit Theorems

(joint work with Michael Drmota)

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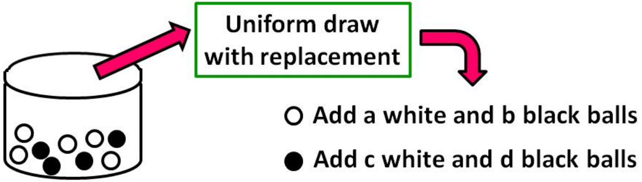

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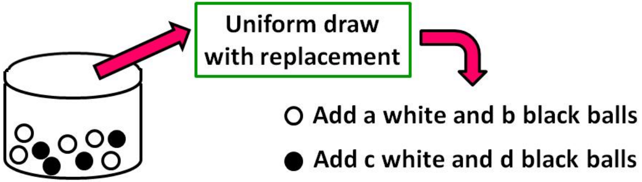

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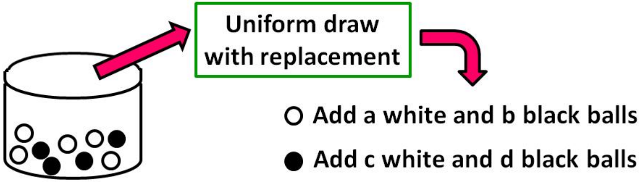

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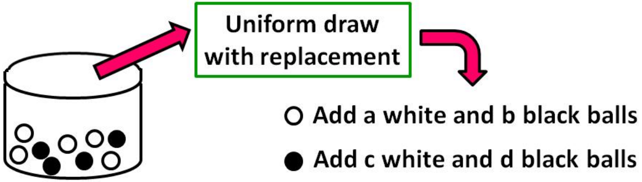

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- Rules for urn evolution $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{Z}$
- After a finite number of successive draws, we reach an absorbing state $s = (j, k) \in \mathcal{S} = \mathbb{N} \times \mathbb{N}$ where the process of successive draws stops when the urn contains exactly j black and k white balls

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- $h_{n,0}(v) = v^n$ and for $(m,n) \notin \mathcal{S}$,

$$h_{n,m}(v) = \frac{n}{n+m} h_{n+a,m+c}(v) + \frac{m}{n+m} h_{n+c,m+d}(v)$$

Three examples from Kuba and Panholzer (2007)

- The pill's problem has transition matrix $\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ and absorbing states $\{(0,n) : n \geq 0\}$
- A variant of the pill's problem has transition matrix $\begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$ with absorbing states $\{(0,n) : n \geq 0\} \cup \{(1,n) : n \geq 0\}$.
- The cannibal urn has transition matrix $\begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$ with absorbing states $\mathcal{S} = \{(0,n) : n \geq 0\} \cup \{(1,n) : n \geq 0\}$

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$$\frac{X_{n,m}}{\frac{n}{m} + \log m} \to \text{Exponential}(1) \qquad (m \to \infty)$$

$$\frac{X_{n,m}}{n} \to \text{Beta}(1,m) \qquad (\text{fixed } m \geq 1,\ n \to \infty)$$

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$$\frac{X_{n,2m}}{\frac{n}{\sqrt{m}} + 2\sqrt{m}} \to R \qquad (m \to \infty \;\; R \text{ has density } 2xe^{-x^2},\, x \geq 0),$$

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$$\frac{X_{n,m} - \mathbb{E}\, X_{n,m}}{\sqrt{\mathbb{V}\mathrm{ar} X_{n,m}}} \to N(0,1) \qquad (m+n\to\infty).$$

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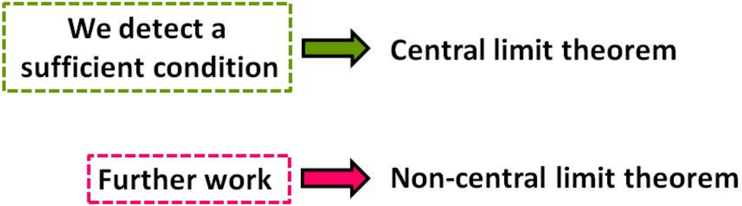

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First we have the system of first order ODE:

$$\frac{dz}{dt} = A(z,w), \quad \frac{dw}{dt} = B(z,w), \quad \frac{dH}{dt} = C(z,w)H + D(z,w;v),$$

where $z = z(t)$, $w = w(t)$, $H = H(t)$ are functions in t.

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$$H(z,w;v) = \exp\left(\int_0^w \frac{C(f(Q_1(z,w),s),s)}{B(f(Q_1(z,w),s),s)}\,ds\right)$$
$$\times \left(\int_0^w \frac{D(f(Q_1(z,w),s),s;v)}{B(f(Q_1(z,w),s),s)} \exp\left(-\int_0^s \frac{C(f(Q_1(z,w),t),t)}{B(f(Q_1(z,w),t),t)}\,dt\right) ds\right)$$

2. Singularity analysis of $H(z,w;v)$

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$$:= \exp\left(K(z,w)\right)\int_0^w \frac{D(f(Q_1(z,w),s),s;v)}{B(f(Q_1(z,w),s),s)} \exp\left(-K(z,s)\right) ds$$
$$= \int_0^w \frac{a(f(Q_1(z,w),s),s;v)\exp\left(-K(z,s)\right)/B(f(Q_1(z,w),s),s)}{(1-b(f(Q_1(z,w),s),s;v))^2}\,ds$$

$$:= \int_0^w \frac{L(z,w,s)}{(1-b(f(Q_1(z,w),s),s;v))^2}\,ds$$

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$$\forall z,w \in \mathbb{R}^+, \quad \frac{\partial f}{\partial s} = \frac{A(f,s)}{B(f,s)} < 0 \quad \Rightarrow \quad f(Q_1(z,w),0) = 1$$

$$\Leftrightarrow \quad z + w \to 1$$

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$$H(z,w;1) = \sum_{n\geq 0, m\geq 1} \binom{n+m}{m} h_{n,m}(1) z^n w^m = \frac{1}{1-z-w} - \frac{1}{1-z}$$

$$\forall z,w \in \mathbb{R}^+, \quad \frac{\partial f}{\partial s} = \frac{A(f,s)}{B(f,s)} < 0 \quad \Rightarrow \quad f(Q_1(z,w),0) = 1$$

$$\Leftrightarrow \quad z + w \to 1$$

$$H(z,w;1) = \frac{\tilde{L}(z,w)}{1-f(Q_1(z,w),0)} + O\left(\log|1-f(Q_1(z,w),0)|\right)$$

2. Singularity analysis of $H(z,w;v)$: when $v=1$

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$H(z,w;1)$ will not get singular if $z+w\neq 1$. But this representation for $H(z,w;1)$ holds if v is close to 1 and $|1-z-w|\geq\delta$ for some $\delta>0$.

2. Singularity analysis of $H(z, w; v)$: when v close to 1

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If z and w satisfy $|1-z-w|<\delta$, for some $\delta>0$, then

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$$H(z, w; v) \sim \frac{N(z, w; v)}{1 - b(f(Q_1(z, w), 0), 0; v)}$$

3. Asymptotics for bivariate generating functions

A generalized case in Pemantle and Wilson's book (2013)

Lemma Let $f(z,w)$ be a bivariate generating function in the form

$$f(z,w) = \frac{N(z,w)}{D(z,w)},$$

where N and D are regular functions such that the system

$$D(z,w) = 0, \quad wD_w(z,w) = \rho z D_z(z,w)$$

has a unique positive and analytic solution $z = z_0(\rho)$, $w = w_0(\rho)$ for $\rho \in [\alpha, \beta]$ such that

- $D_w(z_0(\rho), w_0(\rho)) \neq 0$
- $D(z,w) = 0$ has no other solutions for $|z| \leq z_0(\rho)$, $|w| \leq w_0(\rho)$
- $N(z_0(\rho), w_0(\rho)) \neq 0$

3. Asymptotics for bivariate generating functions (next)

Then we have uniformly for $m/n \in [\alpha, \beta]$

$$[z^n w^m] f(z,w) \sim \frac{N(z_0(m/n), w_0(m/n))}{-z_0(m/n) w_0(m/n) D_z(z_0(m/n), w_0(m/n))} \cdot \frac{z_0(m/n)^{-n} w_0(m/n)^{-m}}{\sqrt{2\pi n \Delta(m/n)}},$$

where

$$\Delta(\rho) = \frac{D_{zz} D_w^2 - 2 D_{zw} D_z D_w + D_{ww} D_z^2}{z D_z^3} + \frac{D_w^2}{z^2 D_z^2} + \frac{D_w}{z w D_z} \bigg|_{z=z_0(\rho), w=w_0(\rho)}$$

3. Asymptotics for bivariate generating functions (proof)

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- Since $D(z,w)=0$ has a unique solution $z=z(w)$ for each w close to real line $[w_0(a), w_0(b)]$, then

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- We fix the ratio $m/n=\rho$ and $h_\rho(w) := -\ln\left(z(w)+w^\rho\right)$

$$[w^{n\rho}z^n]f(z,w) \sim \frac{1}{2\pi i}\int_{|w|=w_0(\rho)} \frac{N(z(w),w)z(w)^{-n}e^{nh(w)}}{-w\cdot z(w)D_z(z(w),w)}\,dw$$
$$\sim \frac{N(z_0(\rho),w_0(\rho))z_0(\rho)^{-n}w_0(\rho)^{-n\rho}}{-w_0(\rho)z_0(\rho)D_z(z_0(\rho),w_0(\rho))\sqrt{2\pi n h''_\rho(w_0(\rho))}}$$

1. $w_0(\rho)$ is the saddle point of $h_\rho(w)$
2. $z_0(\rho) := z(w_0(\rho))$ and $\Delta(\rho) := h''_\rho(w_0(\rho))$
3. $h'_\rho(w) = 0 \quad \Rightarrow \quad wD_w(z,w) = \rho z D_z(z,w)$

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that is uniform in v (for v sufficiently close to 1) and also $z_0(\rho;v)$ and $w_0(\rho;v)$ denote the solutions of the system of equations

$$\begin{cases} b(f(Q(z,w),0),0;v) = 1 \\ z\frac{\partial}{\partial z}b(f(Q(z,w),0),0;v) = \rho w \frac{\partial}{\partial z} b(f(Q(z,w),0),0;v) \end{cases}$$

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If we fix the ratio $\rho = m/n$ the leading asymptotics is

$$z_0(\rho;v)^{-n} w_0(\rho;v)^{-\rho n} = e^{h(\rho;v)n}$$

with $h(\rho;v) = -\log z_0(\rho;v) - \rho \log w_0(\rho;v)$.

5. Quasi-Power Theorem

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$$\mathbb{E}[v^{X_{\rho n,n}}] = \frac{[z^n w^{\rho n}]H(z,w;v)}{[z^n w^{\rho n}]H(z,w;1)} \sim \frac{C(\rho;v)}{C(\rho;1)} e^{(h(\rho;v)-h(\rho;1))n}.$$

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$$\frac{X_{n,m} - \mathbb{E}\, X_{n,m}}{\sqrt{n}} \to \mathcal{N}(0, \sigma^2(m/n))$$

uniformly for $m + n \to \infty$, $m/n \in [\alpha, \beta]$, where

$$\mathbb{E}\, X_{n,m} \sim \mu(m/n)\, n, \qquad \mathbb{V}\mathrm{ar} X_{n,m} \sim \sigma^2(m/n)\, n,$$

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Thank You!