Combinatorial Analysis of Growth Models for Series-Parallel Networks

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Series-parallel networks

Series-parallel networks (SP-nets):

- two-terminal graphs: distinguished vertices source and sink
- generated recursively by two composition operations:
    - **parallel composition:**
      sources and sinks of two SP-nets are merged
    - **series composition:**
      sink of one SP-net merged with source of other SP-net

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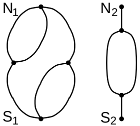

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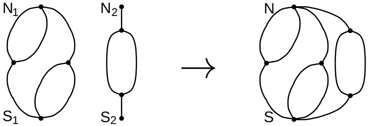

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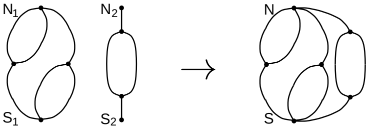

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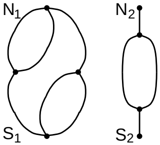

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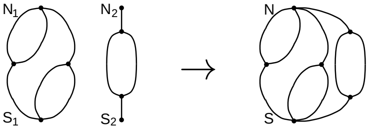

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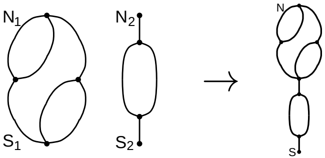

Applications of SP-nets:

- model flow in bipolar networks (electric circuits, etc.)
- computational complexity theory: some in general NP-complete graph problems are solvable in linear time for SP-nets (max. independent set, . . .)

Models for SP-nets/SP-graphs:

various studies of typical behaviour of structural quantities in SP-nets/SP-graphs

- **uniform models:** all SP-nets/SP-graphs of given size are equally likely
  e.g., Bodirsky, Gimenez, Kang, Noy [2007]; Bernasconi, Panagiotou, Steger [2008]; Drmota, Gimenez, Noy [2010,2011]; Hofri, Li, Mahmoud [2016]
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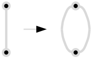
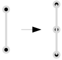
each SP-net can be generated by basic edge-duplication rules:

- parallel doubling: 
- serial doubling: 

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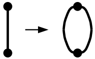
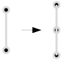
- parallel doubling: 
- serial doubling: 

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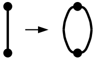
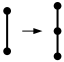
- parallel doubling: 
- serial doubling: 

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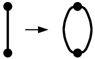
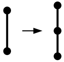
- parallel doubling: 
- serial doubling: 

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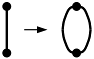
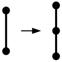
- parallel doubling: 
- serial doubling: 

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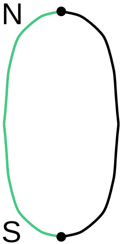

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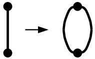
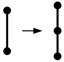
- parallel doubling: 
- serial doubling: 

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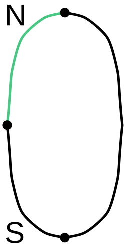

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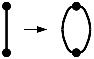
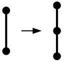
- parallel doubling: 
- serial doubling: 

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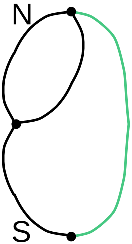

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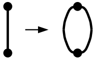
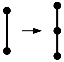
- parallel doubling: 
- serial doubling: 

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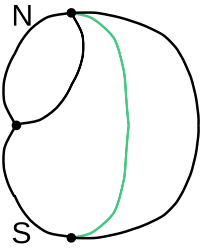

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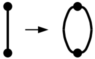
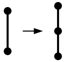
- parallel doubling: 
- serial doubling: 

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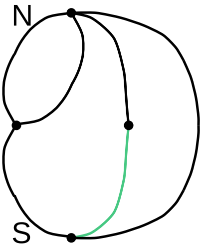

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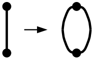
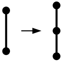
- parallel doubling: 
- serial doubling: 

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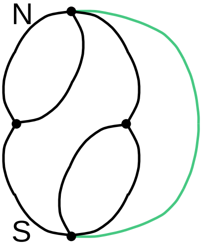

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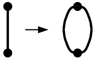
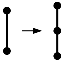
- parallel doubling: 
- serial doubling: 

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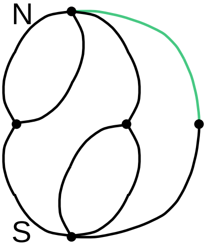

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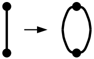
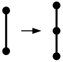
- parallel doubling: 
- serial doubling: 

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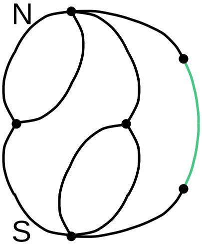

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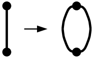
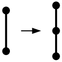
- parallel doubling: 
- serial doubling: 

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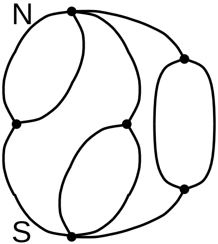

Stochastic growth models for SP-nets

Bernoulli model [Mahmoud, 2013]:

- **step 1:** start with single edge
- **step n:** select edge $e =$ uniformly at random
  - with probability p: parallel duplication of e

  - with probability $q = 1 - p$: serial duplication of e

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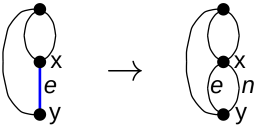

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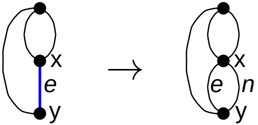

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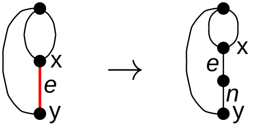

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Binary model [Mahmoud, 2014]:

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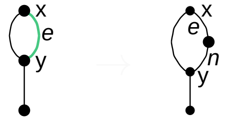

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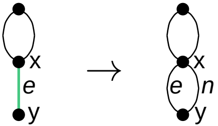

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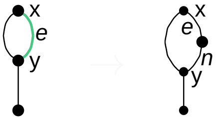

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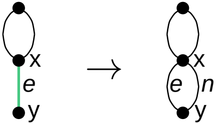

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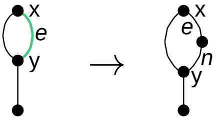

Combinatorial description

Stochastic growth rule

→ "bottom-up" analysis of quantities in SP-net models via Pólya-Eggenberger urn models [Mahmoud, 2013 & 2014]: nodes of small degree, degree of the source, length of random source-to-sink path

Combinatorial description of SP-net models

→ capture growth process via increasing tree models

→ allows combinatorial decomposition of structure

→ "top-down" approach for analysis of quantities in SP-nets by considering parameters in corresponding tree models

→ approach can be extended to further growth rules for SP-nets

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Bernoulli model → edge-coloured recursive trees:

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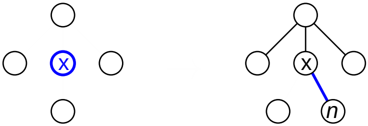

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Bernoulli model $\rightarrow$ edge-coloured recursive trees:

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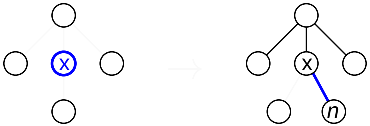
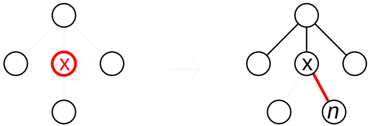
- **step** n: select node ⓧ uniformly at random
  - with probability p: attach node n to x with blue edge



  - with probability $q = 1 - p$: attach node n to x with red edge



Combinatorial description of SP-net models

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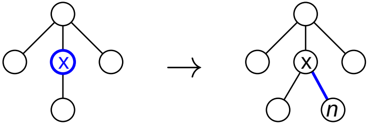

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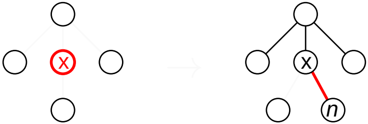

Combinatorial description of SP-net models

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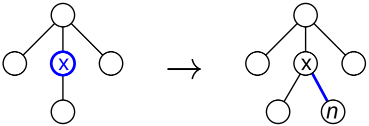

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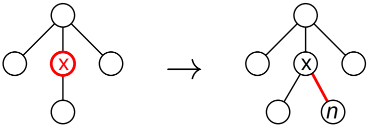
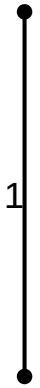


Combinatorial description of SP-net models

Example:

SP-net



edge-coloured recursive tree

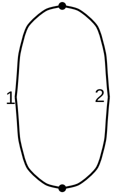
- edge-coloured unordered increasing trees
- parallel duplication $\hat{=}$ blue edge
- serial duplication $\hat{=}$ red edge
- "top-down" decomposition of tree family $\mathcal{T}$:

$$\boxed{\mathcal{T} \;=\; ① \;\dot\cup\; ① \times \textsc{Set}(\{\,|\,\} \times \mathcal{T} \;\dot\cup\; \{\,|\,\} \times \mathcal{T})}$$

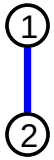
Combinatorial description of SP-net models

Example:

SP-net



edge-coloured recursive tree



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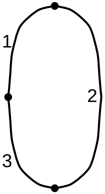
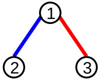
Combinatorial description of SP-net models

Example:

SP-net



edge-coloured recursive tree



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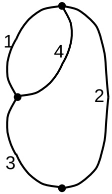
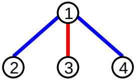
Combinatorial description of SP-net models

Example:

SP-net



edge-coloured recursive tree



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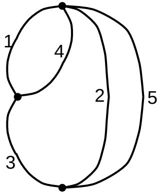
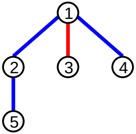
Example:

SP-net



edge-coloured recursive tree



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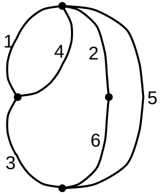
- "top-down" decomposition of tree family $\mathcal{T}$:

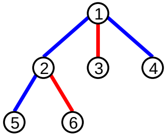
$$\boxed{\mathcal{T} \;=\; \textcircled{1} \;\dot\cup\; \textcircled{1} \times \textsc{Set}\big(\{\,\color{blue}{|}\,\} \times \mathcal{T} \;\dot\cup\; \{\,\color{red}{|}\,\} \times \mathcal{T}\big)}$$

Example:

SP-net



edge-coloured recursive tree



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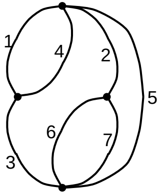
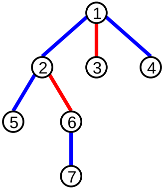
Combinatorial description of SP-net models

Example:

SP-net



edge-coloured recursive tree



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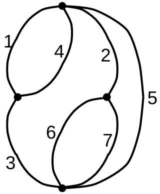
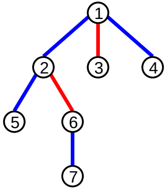
Combinatorial description of SP-net models

Example:

SP-net



edge-coloured recursive tree



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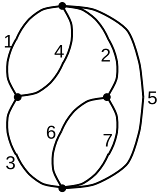
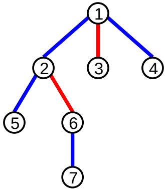
Combinatorial description of SP-net models

Example:

SP-net



edge-coloured recursive tree



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Combinatorial description of SP-net models

Binary model $\rightarrow$ bucket recursive trees:

- nodes in tree can hold up to two labels
- **step** 1: start with single node containing label ①
- **step** n: select label $1 \leq j < n$ uniformly at random
  - if node x containing j is not saturated: label n will be inserted into x

  - if node x containing j is saturated: new node containing label n will be attached to x

Combinatorial description of SP-net models

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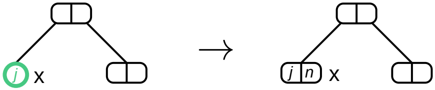

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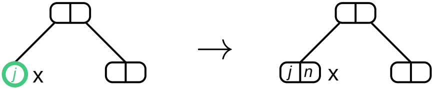

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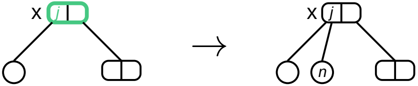
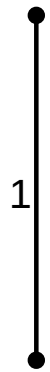


Combinatorial description of SP-net models

Example:

SP-net

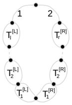


bucket recursive tree

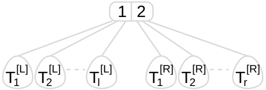
- "top-down" decomposition of bucket rec. tree family $\mathcal{B}$:

$$\boxed{\mathcal{B} = (1) \,\dot{\cup}\, \boxed{1\,|\,2} \times \mathrm{SET}(\mathcal{B}) \times \mathrm{SET}(\mathcal{B})}$$

- tree decomposition reflects subblock-structure



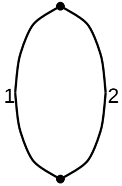
$\longleftrightarrow$



Combinatorial description of SP-net models

Example:

SP-net



bucket recursive tree

$\boxed{1 \mid 2}$

- "top-down" decomposition of bucket rec. tree family $\mathcal{B}$:

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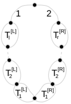

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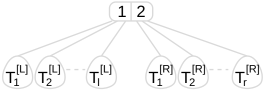
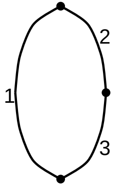


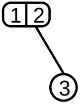
Combinatorial description of SP-net models

Example:

SP-net



bucket recursive tree



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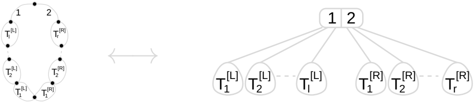
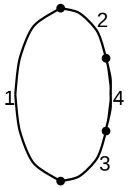


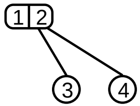
Combinatorial description of SP-net models

Example:

SP-net



bucket recursive tree



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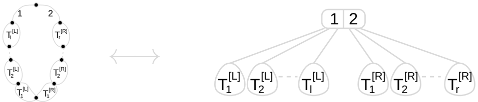
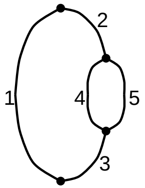


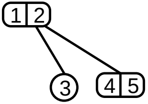
Combinatorial description of SP-net models

Example:

SP-net



bucket recursive tree



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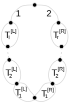

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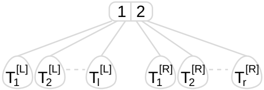
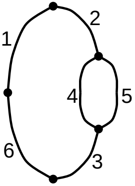


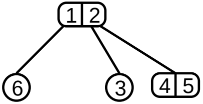
Combinatorial description of SP-net models

Example:

SP-net



bucket recursive tree



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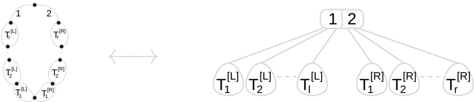
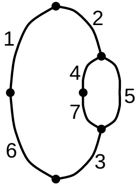
- tree decomposition reflects subblock-structure



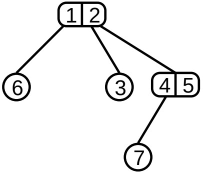
Combinatorial description of SP-net models

Example:

SP-net



bucket recursive tree



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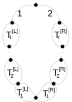
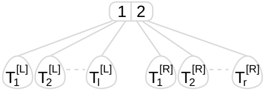


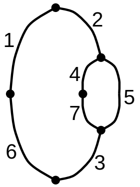
$\longleftrightarrow$



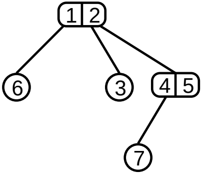
Combinatorial description of SP-net models

Example:

SP-net



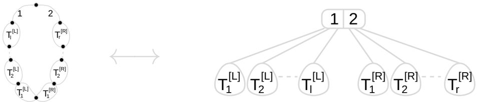
bucket recursive tree



- “top-down” decomposition of bucket rec. tree family $\mathcal{B}$:

$$\boxed{\mathcal{B} \;=\; \textcircled{1} \;\dot{\cup}\; \boxed{1|2} \times \mathrm{SET}(\mathcal{B}) \times \mathrm{SET}(\mathcal{B})}$$

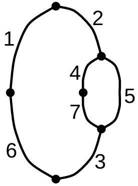
- tree decomposition reflects subblock-structure



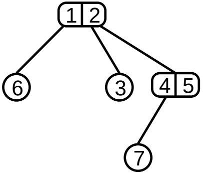
Combinatorial description of SP-net models

Example:

SP-net



bucket recursive tree



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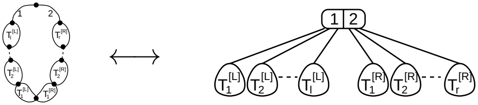
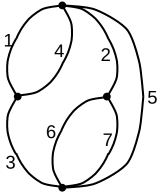
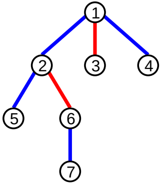


Parameter I: source degree for Bernoulli model

SP-net



edge-coloured recursive tree



Generating functions approach:

- count $\sharp$ of trees of order n with k blue edges and blue subtree has order m
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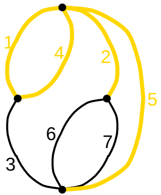
degree of source



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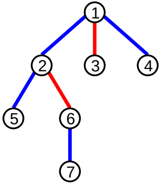

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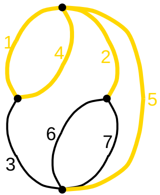
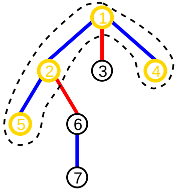


order of blue subtree



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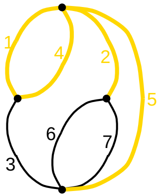

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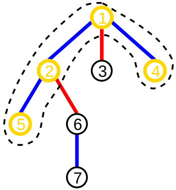

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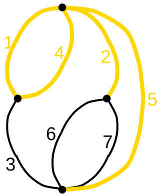

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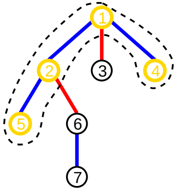

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D_n : source degree of size-n SP-net

Exact probability distribution [Mahmoud, 2013]:

$$\mathbb{P}\{D_n = m\} = \sum_{j=0}^{m-1} \binom{m-1}{j} (-1)^{n+j-1} \binom{p(j+1)-1}{n-1}$$

Theorem (Kuba & Pan, 2016)

Limiting distribution behaviour: $\frac{D_n}{n^p} \xrightarrow{(d)} D$,

with $D = D(p)$ Mittag-Leffler distribution with parameter p

Sequence of moments $\mathbb{E}(D^r) = \frac{r!}{\Gamma(rp+1)}$, *for* $r \geq 0$

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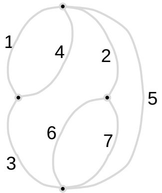
L_n: length of random source-to-sink path in random size-n SP-net

- start at source; at each node x choose one of $d^+(x)$ outgoing edges uniformly at random; until reaching sink

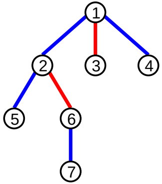
Observations:

- $L_n \stackrel{(d)}{=} L_n^{[left]}$: length of leftmost source-to-sink path

SP-net



edge-coloured recursive tree



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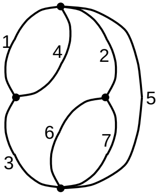

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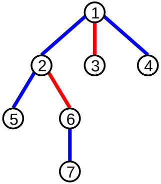

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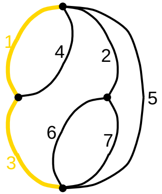

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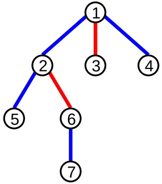

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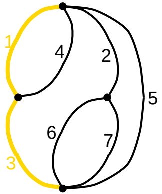

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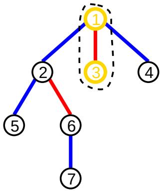

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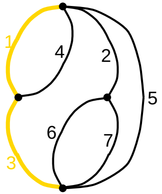

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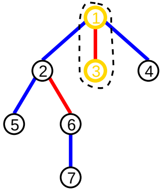

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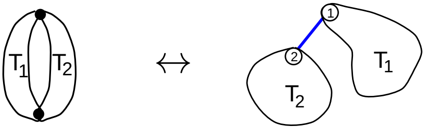
Source-to-sink paths according to first edge-duplication:

parallel duplication: $paths(T) = paths(T_1) + paths(T_2)$

T :



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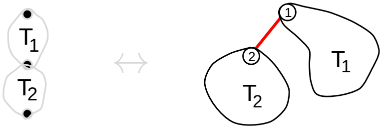

P_n: number of source-to-sink paths in random size-n SP-net

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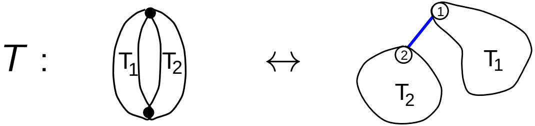

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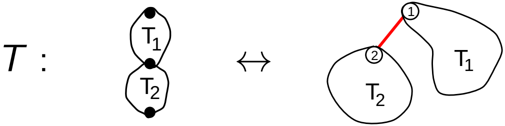

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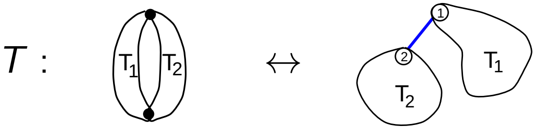

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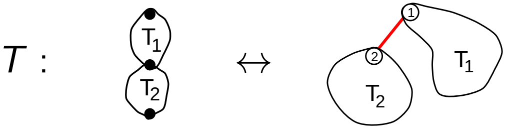

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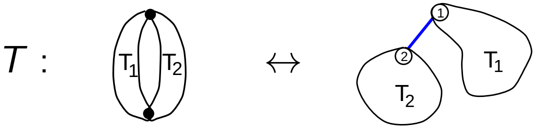

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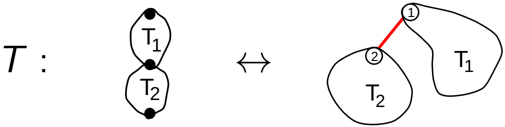

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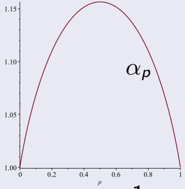
Theorem (Kuba & Pan, 2016)

Asymptotic behaviour of expectation $\mathbb{E}(P_n)$:

$$\mathbb{E}(P_n) = \frac{1}{1-p} \cdot \alpha_p^n + R_p(n),$$

$$\textit{where } \alpha_p = \begin{cases} \frac{1}{1-\left(\frac{p}{1-p}\right)^{\frac{1}{1-2p}}}, & \textit{for } p \neq \frac{1}{2}, \\ \frac{1}{1-e^{-2}}, & \textit{for } p = \frac{1}{2} \end{cases}$$



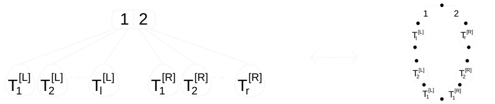
$$\textit{and } R_p(n) = \begin{cases} -\frac{1-2p}{p\Gamma(2p)} n^{2p-1} + \mathcal{O}(n^{2(2p-1)}), & \textit{for } 0 < p < \frac{1}{2}, \\ -\frac{2}{\log n} + \mathcal{O}(\frac{1}{\log^2 n}), & \textit{for } p = \frac{1}{2}, \\ -\frac{2p-1}{1-p} + \mathcal{O}(n^{1-2p}), & \textit{for } \frac{1}{2} < p < 1 \end{cases}$$

Par. IV: length of random path for Binary mod.

L_n: length of random source-to-sink path in random size-n SP-net

Observations:

- $L_n \stackrel{(d)}{=} L_n^{[left]}$: length of leftmost source-to-sink path
- subblock structure $\rightarrow$ recursive description

$T:$ 

$\rightarrow \quad length(T) = 1 + \underbrace{length(T_1^{[L]}) + \cdots + length(T_\ell^{[L]})}_{\text{sum of lengths in left forest}}$

Generating functions approach:

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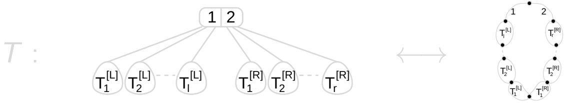

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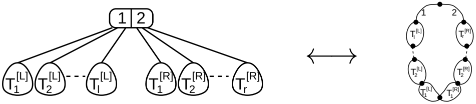

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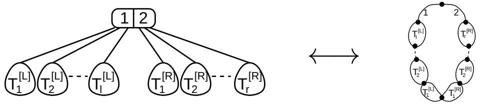
T :



$$\rightarrow \qquad length(T) = 1 + \underbrace{length(T_1^{[L]}) + \cdots + length(T_\ell^{[L]})}_{\text{sum of lengths in left forest}}$$

Generating functions approach:

- apply combinatorial decomposition
- g.f. $F(z,v) \rightarrow$ non-linear DEQ: $F''(z,v) = \frac{v}{1-z} e^{F(z,v)}$
- apply method of moments

Par. IV: length of random path for Binary mod.

L_n: length of random source-to-sink path in random size-n SP-net

Expectation [Mahmoud, 2013]:

$$\mathbb{E}(L_n) = n\left(\frac{3+\sqrt{5}}{2\sqrt{5}}\binom{n+\frac{\sqrt{5}}{2}-\frac{3}{2}}{n} - \frac{3-\sqrt{5}}{2\sqrt{5}}\binom{n-\frac{\sqrt{5}}{2}-\frac{3}{2}}{n}\right) \sim \frac{1+\sqrt{5}}{2\sqrt{5}}\frac{n^{\frac{\sqrt{5}-1}{2}}}{\Gamma(\frac{\sqrt{5}-1}{2})}$$

Theorem (Kuba & Pan, 2016)

Limiting distribution behaviour: $\frac{L_n}{n^{\phi}} \xrightarrow{(d)} L, \quad \phi = \frac{\sqrt{5}-1}{2},$

with L characterized by sequence of r-th integer moments:

$$\mathbb{E}(L^r) = \frac{r!\cdot c_r}{\Gamma(r\phi+1)}, \quad r \geq 0,$$

where sequence $(c_r)_r$ satisfies recurrence $c_0 = 1$, $c_1 = \frac{3+\phi}{5}$,

$$c_r = \frac{1}{\phi(r-1)((r+1)\phi+1)}\sum_{k=1}^{r-1}(k\phi+1)c_k c_{r-k}, \quad \textit{for } r \geq 2$$

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Further Parameters for Binary model

Theorem (Kuba & Pan, 2016)

D_n: *degree of sink in random size-n SP-net*

Limiting distribution behaviour: $\frac{D_n}{n^{\sqrt{2}-1}} \xrightarrow{(d)} D$,

$$\mathbb{E}(D^r) = \frac{r!(r(\sqrt{2}-1)+1)c_r}{\Gamma(r(\sqrt{2}-1)+1)}, \quad r \geq 0,$$

where sequence $(c_r)_{r\geq 0}$ satisfies "convolution-type recurrence"

Theorem (Kuba & Pan, 2016)

P_n: *number of source-to-sink paths in random size-n SP-net*

Expectation: $\mathbb{E}(P_n) = \frac{2}{\rho^n} \cdot \left(1 - \frac{\rho^2}{(\rho-1)^2(n-1)(n-2)} + \mathcal{O}\left(\frac{\log n}{n^4}\right)\right)$,

with $\rho \approx 0.89\ldots$

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Relations to other work

- **Increasing diamonds:**
  - SP-net model introduced by [Bodini, Dien, Fontaine, Genitrini and Hwang, 2016]
  - different structure, study yields second-order non-linear DEQs
- **Mittag-Leffler limiting distributions** in comb. contexts:
  - triangular balanced urn models [Janson, 2010]
  - extra clustering model for animal grouping [Drmota, Fuchs and Lee, 2015]
- **Recurrences of "convolution type":**
  similar to ones studied via Psi-series method in [Chern, Fernández-Camacho, Hwang and Martínez, 2014]
- **Top-down decomposition:**
  maybe alternative characterizations via contraction method, e.g., [Neininger and Rschendorf, 2004]

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Outlook

Study further quantities via combinatorial decomposition:

- number of ancestors of nodes/edges
- number of descendants of nodes/edges
- node degrees
- number of spanning trees
- number of independent sets

Further stochastic SP-net models with comb. description:

- b-ary saturation model for SP-nets $\leftrightarrow$ bucket recursive trees with bucket-size $b \geq 2$
- "preferential edge-duplication rule" for Bernoulli model $\leftrightarrow$ edge-coloured plane increasing trees

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