A half-normal distribution scheme for generating functions and the unexpected behavior of Motzkin paths

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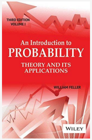
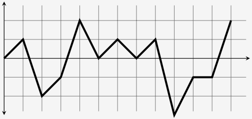
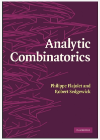
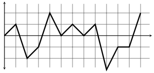






What is a lattice path?



Definition

- **Step set:** $\mathcal{S} = \{(1, b_1), \ldots, (1, b_m)\} \subset \mathbb{Z}^2$
- ***n*-step lattice path:** Sequence of vectors $(v_1, \ldots, v_n) \in \mathcal{S}^n$

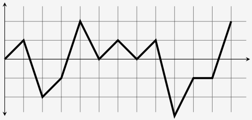
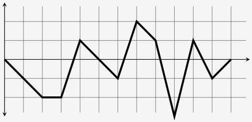
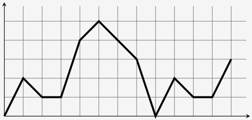
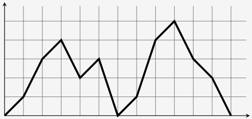
Weights

- For $\mathcal{S} = \{-c, \ldots, d\}$ define $\Pi = \{p_{-c}, \ldots, p_d\}$
- **Jump polynomial:** $P(u) = \sum_{i=-c}^{d} p_i u^i$
- **Drift:** $\delta = P'(1)$

Examples

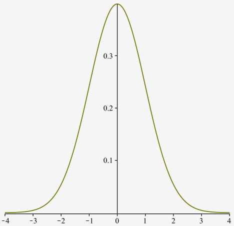
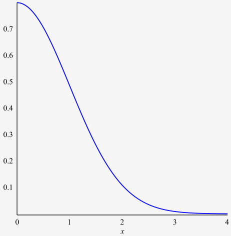
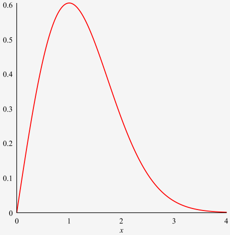
- **Dyck path/Random walk:** $P(u) = p_{-1}u^{-1} + p_1 u^1$
- **Motzkin walk:** $P(u) = p_{-1}u^{-1} + p_0 + p_1 u^1$

Terminology of directed paths

	ending anywhere	ending at 0
unconstrained (on $\mathbb{Z}$)	 walk/path ($\mathcal{W}$)	 bridge ($\mathcal{B}$)
constrained (on $\mathbb{Z}_+$)	 meander ($\mathcal{M}$)	 excursion ($\mathcal{E}$)

One-dimensional objects

Probability distributions

	Normal	**Half-normal**	**Rayleigh**
			
Support	$x \in \mathbb{R}$	$x \in \mathbb{R}_{\geq 0}$	$x \in \mathbb{R}_{\geq 0}$
PDF	$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$	$\sqrt{\frac{2}{\pi}} \exp\left(-\frac{x^2}{2}\right)$	$x \exp\left(-\frac{x^2}{2}\right)$
Mean	0	$\sqrt{\frac{2}{\pi}}$	$\sqrt{\frac{\pi}{2}}$
Variance	1	$1 - \frac{2}{\pi}$	$2 - \frac{\pi}{2}$

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	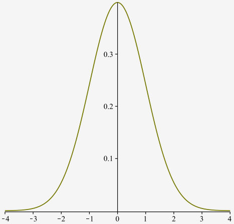	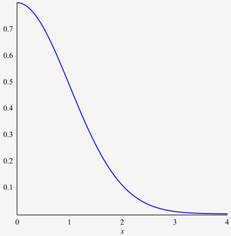	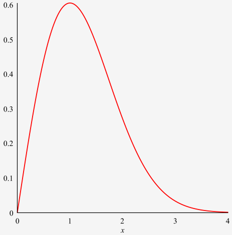
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Sign changes of Motzkin walks

Motzkin walk

- Unconstrained walk generated from
- Step polynomial $P(u) = p_{-1}u^{-1} + p_0 + p_1 u$ with
- $p_{-1}, p_0, p_1 \in \mathbb{R}_+$.

Sign changes

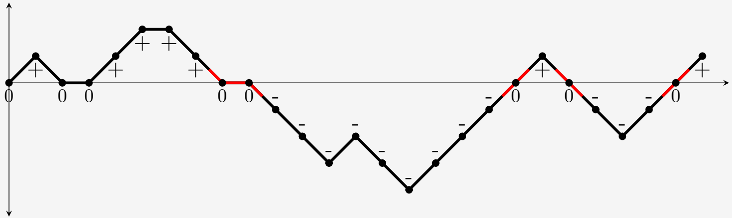
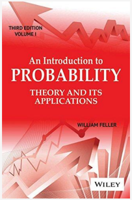


Figure: A signed Motzkin walk with 4 sign changes

Feller's Caveat

"We shall encounter theoretical conclusions which not only are unexpected but actually come as a shock to intuition and common sense."

(William Feller, An Introduction to Probability Theory and its Applications, Volume 1, Fluctuations in Coin Tossing and Random Walks)



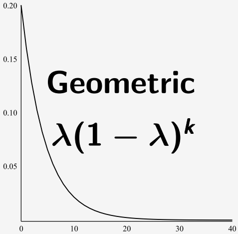
Limit laws for the number of sign changes

Theorem [Extension: Feller, Ch. III.5]

Let X_n denote the number of sign changes of Motzkin walks of length n. Let $\delta = P'(1)$ be the drift.

1. For $\delta \neq 0$ we get convergence to a **geometric distribution**:
$$X_n \xrightarrow{d} \text{Geom}(\lambda) \quad \text{with} \quad \lambda = \begin{cases} \frac{p_1}{p_{-1}}, & \text{for } \delta < 0, \\ \frac{p_{-1}}{p_1}, & \text{for } \delta > 0; \end{cases}$$

2. For $\delta = 0$ we get convergence to a **half-normal distribution**:
$$\frac{X_n}{\sqrt{n}} \xrightarrow{d} \mathcal{H}(\sigma) \quad \text{with} \quad \sigma = \frac{1}{2}\sqrt{\frac{P''(1)}{P(1)}}.$$



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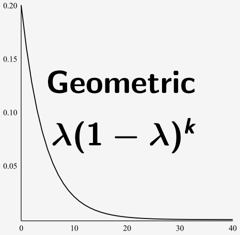
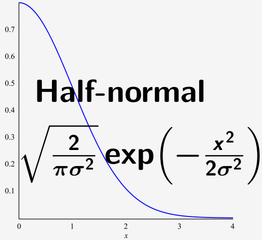
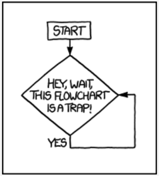




What is a scheme?

Scheme
an organized plan for doing something, especially something dishonest or illegal that will bring a good result for you

(Cambridge Dictionary)



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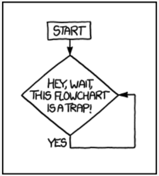

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Schema

a drawing that represents an idea or theory and makes it easier to understand

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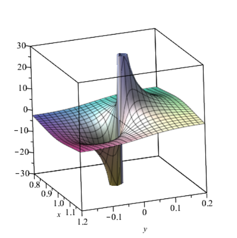

Strategy

1. **Count:**
   Generating function $F(z) = \sum_{n \geq 0} f_n z^n$
   $f_n \dots$ # objects of size n
2. **Mark:**
   Bivariate generating function $F(z, u) = \sum_{n,k \geq 0} f_{n,k} z^n u^k$
   $f_{n,k} \dots$ # objects of size n with k occurrences of a certain property
3. **Analyze:**
   1. *Algebraic:* Structure of $F(z, u)$, non-negative coefficients, ...
   2. *Analytic:* Convergence, singularities, ...

We get:

- **Probability distribution** of a marked parameter for large n
- Moments
- Asymptotic behavior
- ...

Examples of schemes

Many examples in "Analytic Combinatorics" Chapter IX. Multivariate Asymptotics and Limit Laws [Flajolet–Sedgewick '09].

- **Discrete limit laws**
- **Normal distribution**
  - Central limit theorems [Bender '73], [Bender–Richmond '83], [Flajolet–Soria '90], [Hwang '94], ...
  - Quasi-powers Theorem [Hwang '98]
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Family of three limit laws: [Drmota–Soria '97]

1. Rayleigh distribution
2. Normal distribution
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Family of three limit laws: [Drmota–Soria '97]

1. Rayleigh distribution
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Goal: Extend this family by a half-normal distribution [W '16]

Images and preimages in random mappings [Drmota–Soria, '97]

Hypothesis [H]. Let $F(z,u) = \sum_{n,k} f_{nk} z^n u^k$ be a power series in two variables with nonnegative coefficients $f_{nk} \geq 0$ such that $f(z,1)$ has a radius of convergence of $\rho > 0$.

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We suppose that $F(z,u)$ has the local representation

$$\frac{1}{F(z,u)} = g(z,u) + h(z,u)\sqrt{1 - \frac{z}{\rho(u)}} \tag{25}$$

for $|u-1| < \varepsilon$ and $|z - \rho(u)| < \varepsilon$, $\arg(z - \rho(u)) \neq 0$, where $\varepsilon > 0$ is some fixed real number, and $g(z,u)$, $h(z,u)$, and $\rho(u)$ are analytic functions.

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Furthermore, these functions satisfy $g(\rho,1) = 0$, $h(\rho,1) > 0$, and $\rho(1) = \rho$.

In addition, $z = \rho(u)$ is the only singularity on the circle of convergence $|z| = |\rho(u)|$ for $|u-1| < \varepsilon$ and $F(z,u)$, can be analytically continued to a region $|z| < \frac{\varepsilon}{2} + \delta, |u| < 1 + \delta, |u-1| > \frac{\varepsilon}{2}$ for some $\delta > 0$.

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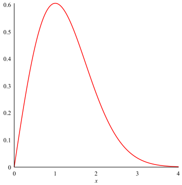
Extension to the half-normal distribution

Theorem [Drmota–Soria, '97]

*Let $F(z,u)$ be a bivariate generating function satisfying [H] . If $\boldsymbol{\rho(u) = \rho = }$ **const** for $|u-1| < \varepsilon$ and $g_u(\rho,1) < 0$,*
then the sequence of random variables X_n defined by

$$\mathbb{P}[X_n = k] = \frac{[z^n u^k]\, F(z,u)}{[z^n]\, F(z,1)}$$



has a **Rayleigh** *limiting distribution; i.e.*

$$\frac{X_n}{\sqrt{n}} \xrightarrow{d} \mathcal{R}(\lambda),$$

where $\lambda = \frac{h(\rho,1)^2}{2g_u(\rho,1)^2}$ and $\mathcal{R}(\lambda)$ has density $\lambda x \exp\left(-\frac{\lambda x^2}{2}\right)$ for $x \geq 0$.

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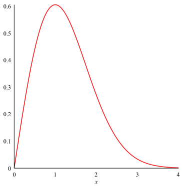

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Expected value and variance are given by

$$\mathbb{E}X_n = \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{\lambda}} \sqrt{n} + \mathcal{O}(1) \quad \text{and} \quad \mathbb{V}X_n = \left(2 - \frac{\pi}{2}\right) \frac{n}{\lambda} + \mathcal{O}(\sqrt{n}).$$

Moreover, we have uniformly for all $k \geq 0$ *the local law*

$$\mathbb{P}[X_n = k] = \frac{\lambda k}{n} \exp\left(-\frac{\lambda k^2}{2n}\right) + \mathcal{O}((k+1)n^{-3/2}) + \mathcal{O}(n^{-1}).$$

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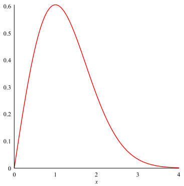

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where $\lambda = \frac{h(\rho,1)^2}{2g_u(\rho,1)^2}$ and $\mathcal{R}(\lambda)$ has density $\lambda x \exp\left(-\frac{\lambda x^2}{2}\right)$ for $x \geq 0$.

Extension to the half-normal distribution

Theorem [DrSo, '97] $\to$ Half Normal Theorem [W, '16]

*Let $F(z,u)$ be a bivariate generating function satisfying [H']. If $\boldsymbol{\rho(u) = \rho =}$ **const** for $|u-1| < \varepsilon$ and $g_u(\rho,1) = g_{uu}(\rho,1) = h(\rho,1) = 0$, $g_z(\rho,1) \neq 0$, and $h_u(\rho,1) \neq 0$, then the sequence of random variables X_n defined by*

$$\mathbb{P}[X_n = k] = \frac{[z^n u^k]\, F(z,u)}{[z^n]\, F(z,1)}$$

has a **half-normal** *limiting distribution; i.e.*

$$\frac{X_n}{\sqrt{n}} \xrightarrow{d} \mathcal{H}(\sigma),$$

where $\sigma = \sqrt{2\frac{h_u(\rho,1)}{\rho g_z(\rho,1)}}$ and $\mathcal{H}(\sigma)$ has density $\sqrt{\frac{2}{\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$ for $x \geq 0$.

Extension to the half-normal distribution

Half Normal Theorem [W, '16]

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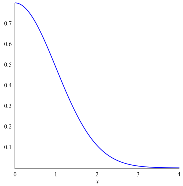

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where $\sigma = \sqrt{2}\frac{h_u(\rho,1)}{\rho g_z(\rho,1)}$ and $\mathcal{H}(\sigma)$ has density $\sqrt{\frac{2}{\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right)$ for $x \geq 0$. Expected value and variance are given by

$$\mathbb{E}X_n = \sqrt{\frac{2}{\pi}}\sigma\sqrt{n} + \mathcal{O}(1) \quad \text{and} \quad \mathbb{V}X_n = \left(1 - \frac{2}{\pi}\right)\sigma^2 n + \mathcal{O}(\sqrt{n}).$$

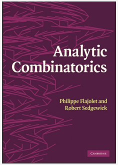
Moreover, we have uniformly for all $k \geq 0$ the local law

$$\mathbb{P}[X_n = k] = \frac{1}{\sigma}\sqrt{\frac{2}{\pi n}}\exp\left(-\frac{k^2/n}{2\sigma^2}\right) + \mathcal{O}\left(kn^{-3/2}\right) + \mathcal{O}(n^{-1}).$$

Sketch of the proof

Technique: Pointwise convergence of characteristic function

$$\mathbb{E}[e^{itX_n/\sqrt{n}}] = \frac{[z^n]F(z, e^{\frac{it}{\sqrt{n}}})}{[z^n]F(z,1)} \overset{n\to\infty}{\to} \varphi_{\mathcal{H}}\left(\frac{\sqrt{2}h_u(\rho,1)}{\rho g_z(\rho,1)} t\right)$$



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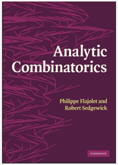

Necessary steps:

- Contour integration to extract asymptotic coefficients
- Bound and estimate gamma function-like integrals

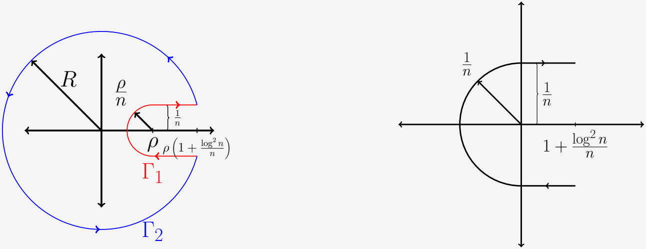

Figure: Hankel contour decomposition (left), and contour at singularity ρ (right).

Returns to zero

Definition

- A *return to zero* is a vertex of a path of altitude 0 whose abscissa is positive.
- An *arch* is a bridge of size > 0 whose only contact with the x-axis is at its end points.

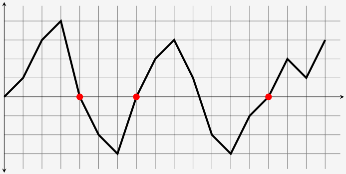

Figure: A walk with 3 returns to zero

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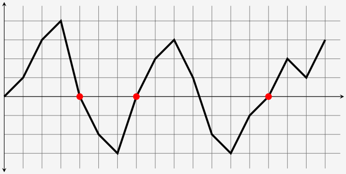
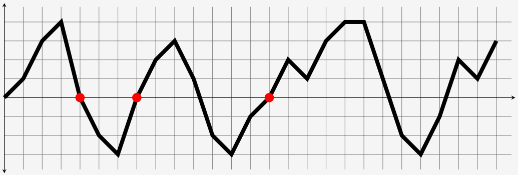


Figure: A walk with 3 returns to zero

Results for directed walks

- $P(u) = p_{-1}u^{-1} + p_0 + p_1 u$
- Walks: $W(z) = \frac{1}{1-zP(1)}$
- Bridges: $B(z)$ known from [Banderier–Flajolet, '02] (Square-root singularity)

Construction of the generating function for returns to zero



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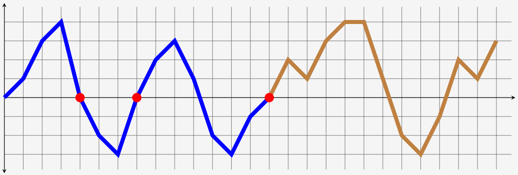
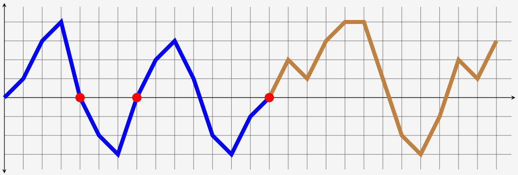


Generating function of Bridges

$$B(z) = \frac{1}{1 - A(z)}, \qquad B(z,u) = \frac{1}{1 - uA(z)}$$

Construction of the generating function for returns to zero



Generating function of Bridges

$$B(z) = \frac{1}{1 - A(z)}, \qquad B(z,u) = \frac{1}{1 - uA(z)}$$

Generating function of the Tail

$$T(z) = \frac{W(z)}{B(z)}$$

Generating function of Walks

$$W(z,u) = \frac{1}{1 - uA(z)} T(z) = \frac{W(z)}{u + (1-u)B(z)}$$

Limit laws for the number of returns to zero

Walks of length n having k returns to zero

$$\mathbb{P}[X_n = k] = \mathbb{P}[\text{size} = n,\ \#\text{returns to zero} = k] = \frac{[z^n u^k]W(z,u)}{[z^n]W(z,1)}$$

Theorem [Extension: Feller, Ch. III, Problems 9-10]

Let X_n denote the number of returns to zero of unconstrained walks of length n. Let $\delta = P'(1)$ be the drift.

1. For $\delta \neq 0$ we get convergence to a geometric distribution:
$$X_n \xrightarrow{d} \text{Geom}\left(\frac{|p_1 - p_{-1}|}{P(1)}\right);$$
2. For $\delta = 0$ we get convergence to a half-normal distribution:
$$\frac{X_n}{\sqrt{n}} \xrightarrow{d} \mathcal{H}\left(\sqrt{\frac{P(1)}{P''(1)}}\right).$$

Proof sketch

$$W(z,u) = \frac{W(z)}{u + (1-u)B(z)} = \frac{1}{1 - zP(1)} \cdot \frac{1}{u + (1-u)B(z)}$$

1. $\delta \neq 0$: **Geometric distribution:** Geom $\left(\frac{1}{B(1/P(1))}\right)$
   Dominant singularity $\frac{1}{P(1)}$ and second factor analytic

$$[z^n]W(z,u) = \frac{1}{B\left(1/P(1)\right)} \frac{P(1)^n}{1 - u\left(1 - \frac{1}{B(1/P(1))}\right)} + o(P(1)^n).$$

2. $\delta = 0$: **Half-normal distribution:** $\mathcal{H}\left(\sqrt{\frac{P(1)}{P''(1)}}\right)$

Proof sketch

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$$[z^n]W(z,u) = \frac{1}{B\left(1/P(1)\right)} \frac{P(1)^n}{1 - u\left(1 - \frac{1}{B(1/P(1))}\right)} + o(P(1)^n).$$

2. $\delta = 0$: **Half-normal distribution:** $\mathcal{H}\left(\sqrt{\frac{P(1)}{P''(1)}}\right)$
   Both factors singular at $\rho := \frac{1}{P(1)}$ and we get for $z \to \rho$ and $u \to 1$

$$\frac{1}{W(z,u)} = g(z,u) \qquad + \qquad \text{h(z,u)}\sqrt{1 - \frac{z}{\rho}}$$

$$= \left(1 - \frac{z}{\rho}\right) u \qquad + \qquad C(1-u)\sqrt{1 - \frac{z}{\rho}} + \mathcal{O}\left(\left(1 - \frac{z}{\rho}\right)(1-u)\right)$$

Proof sketch

$$W(z,u) = \frac{W(z)}{u+(1-u)B(z)} = \frac{1}{1-zP(1)} \cdot \frac{1}{u+(1-u)B(z)}$$

Conditions leading to a half-normal distribution

- $g(\rho, 1) = 0$
- $g_u(\rho, 1) = 0$
- $g_{uu}(\rho, 1) = 0$
- $g_z(\rho, 1) \neq 0$
- $h(\rho, 1) = 0$
- $h_u(\rho, 1) \neq 0$

2 $\delta = 0$: **Half-normal distribution:** $\mathcal{H}\left(\sqrt{\frac{P(1)}{P''(1)}}\right)$

Both factors singular at $\rho := \frac{1}{P(1)}$ and we get for $z \to \rho$ and $u \to 1$

$$\frac{1}{W(z,u)} = g(z,u) + \text{h(z,u)}\sqrt{1-\frac{z}{\rho}}$$

$$= \left(1-\frac{z}{\rho}\right)u + C(1-u)\sqrt{1-\frac{z}{\rho}} + \mathcal{O}\left(\left(1-\frac{z}{\rho}\right)(1-u)\right)$$

Height of Motzkin walks

Motzkin walk

- Unconstrained walk generated from
- Step polynomial $P(u) = p_{-1}u^{-1} + p_0 + p_1 u$ with
- $p_{-1}, p_0, p_1 \in \mathbb{R}_+$.

Height

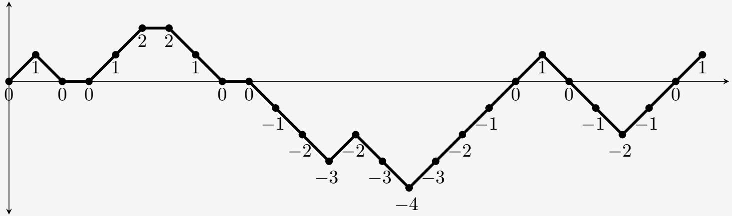

Figure: A Motzkin walk of height 2

Limit laws for the height of Motzkin walks

Theorem [Extension: Feller, Ch. III.7, Theorem 1]

Let X_n denote the height of Motzkin walks of length n.
Depending on the drift $\delta = P'(1)$ we get:

$\underline{\delta < 0}$ $\underline{\delta = 0}$ $\underline{\delta > 0}$

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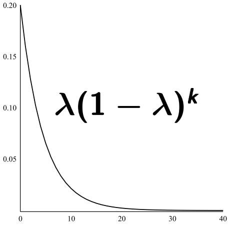
Geometric



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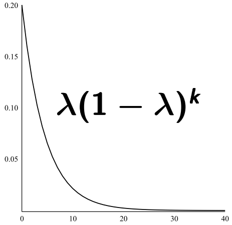
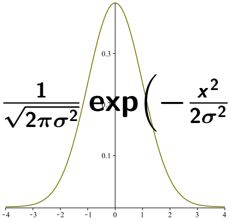
Depending on the drift $\delta = P'(1)$ we get:

$\underline{\delta < 0}$	$\underline{\delta = 0}$	$\underline{\delta > 0}$
Geometric		**Normal**



$$\lambda(1-\lambda)^k$$



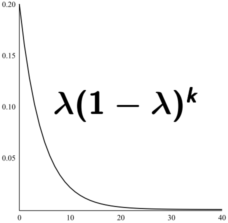
$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

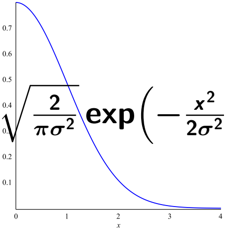
Limit laws for the height of Motzkin walks

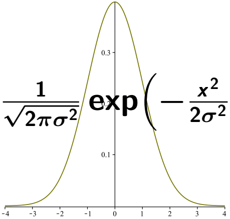
Theorem [Extension: Feller, Ch. III.7, Theorem 1]

Let X_n denote the height of Motzkin walks of length n.
Depending on the drift $\delta = P'(1)$ we get:

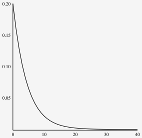
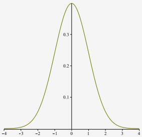
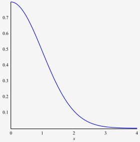
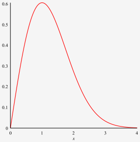
$\delta < 0$	$\delta = 0$	$\delta > 0$
Geometric	**Half-normal**	**Normal**
$\lambda(1-\lambda)^k$	$\sqrt{\frac{2}{\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right)$	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right)$







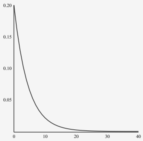
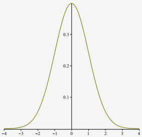
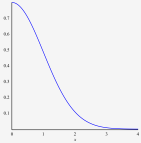
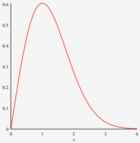
Results for Motzkin paths

	Geometric	**Normal**	**Half-normal**	**Rayleigh**
				
PDF	$\lambda(1-\lambda)^k$	$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$\sqrt{\frac{2}{\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$	$\frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)$
Mean	$\frac{1-\lambda}{\lambda}$	μ	$\sigma\sqrt{\frac{2}{\pi}}$	$\sigma\sqrt{\frac{\pi}{2}}$
Var	$\frac{1-\lambda}{\lambda^2}$	σ^2	$\sigma^2\left(1-\frac{2}{\pi}\right)$	$\sigma^2\left(2-\frac{\pi}{2}\right)$

drift	**returns to zero**	**sign changes**	**height**
$\delta < 0$	$\mathcal{G}\left(\frac{p_{-1}-p_1}{P(1)}\right)$	$\mathcal{G}\left(\frac{p_1}{p_{-1}}\right)$	$\mathcal{G}\left(\frac{p_1}{p_{-1}}\right)$
$\delta = 0$	$\mathcal{H}\left(\sqrt{\frac{P(1)}{P''(1)}}\right)$	$\mathcal{H}\left(\frac{1}{2}\sqrt{\frac{P''(1)}{P(1)}}\right)$	$\mathcal{H}\left(\sqrt{\frac{P''(1)}{P(1)}}\right)$
$\delta > 0$	$\mathcal{G}\left(\frac{p_1-p_{-1}}{P(1)}\right)$	$\mathcal{G}\left(\frac{p_{-1}}{p_1}\right)$	$\mathcal{N}(0,1)$

Table: Limit laws for Motzkin paths ($P(u) = \frac{p_{-1}}{u} + p_0 + p_1 u$) after proper rescaling.

Results for Motzkin paths

	Geometric	**Normal**	**Half-normal**	**Rayleigh**
				
PDF	$\lambda(1-\lambda)^k$	$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$\sqrt{\frac{2}{\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$	$\frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)$
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Var	$\frac{1-\lambda}{\lambda^2}$	σ^2	$\sigma^2\left(1-\frac{2}{\pi}\right)$	$\sigma^2\left(2-\frac{\pi}{2}\right)$

drift	**returns to zero**	**sign changes**	**height**
$\delta < 0$	$\mathcal{G}\left(\frac{p_{-1}-p_1}{P(1)}\right)$	$\mathcal{G}\left(\frac{p_1}{p_{-1}}\right)$	$\mathcal{G}\left(\frac{p_1}{p_{-1}}\right)$
$\delta = 0$	$\mathcal{H}\left(\sqrt{\frac{P(1)}{P''(1)}}\right)$	$\mathcal{H}\left(\frac{1}{2}\sqrt{\frac{P''(1)}{P(1)}}\right)$	$\mathcal{H}\left(\sqrt{\frac{P''(1)}{P(1)}}\right)$
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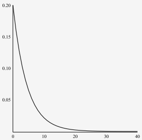
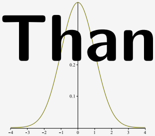
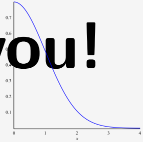
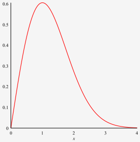
Results for Motzkin paths

Thank you!

	Geometric	**Normal**	**Half-normal**	**Rayleigh**
				
PDF	$\lambda(1-\lambda)^k$	$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$\sqrt{\frac{2}{\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$	$\frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)$
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Var	$\frac{1-\lambda}{\lambda^2}$	σ^2	$\sigma^2\left(1-\frac{2}{\pi}\right)$	$\sigma^2\left(2-\frac{\pi}{2}\right)$

drift	**returns to zero**	**sign changes**	**height**
$\delta < 0$	$\mathcal{G}\left(\frac{p_{-1}-p_1}{P(1)}\right)$	$\mathcal{G}\left(\frac{p_1}{p_{-1}}\right)$	$\mathcal{G}\left(\frac{p_1}{p_{-1}}\right)$
$\delta = 0$	$\mathcal{H}\left(\sqrt{\frac{P(1)}{P''(1)}}\right)$	$\mathcal{H}\left(\frac{1}{2}\sqrt{\frac{P''(1)}{P(1)}}\right)$	$\mathcal{H}\left(\sqrt{\frac{P''(1)}{P(1)}}\right)$
$\delta > 0$	$\mathcal{G}\left(\frac{p_1-p_{-1}}{P(1)}\right)$	$\mathcal{G}\left(\frac{p_{-1}}{p_1}\right)$	$\mathcal{N}(0,1)$

Table: Limit laws for Motzkin paths $\left(P(u) = \frac{p_{-1}}{u} + p_0 + p_1 u\right)$ after proper rescaling.

Backup

Backup slides

Lemmas

Lemma

Let γ be the Hankel contour starting from "$+e^{2\pi i}\infty$", passing around 0 and tending to $+\infty$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{-z}}{z+is\sqrt{-z}}\,dz = \varphi_{\mathcal{H}}\left(\sqrt{2}s\right),$$

where

$$\varphi_{\mathcal{H}}(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{itz}e^{-z^2/2}\,dz,$$

denotes the characteristic function of the Half-normal distribution.

Lemma

Let γ be as in Lemma **??**. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{-s\sqrt{-z}-z}}{\sqrt{-z}}\,dz = \frac{1}{\sqrt{\pi}}e^{-s^2/4}.$$

Marking the height

$$F(z,u) = \frac{1}{1 - zP(1)} \frac{1 - \frac{p_1}{p_{-1}} u_1(z)}{1 - u \frac{p_1}{p_{-1}} u_1(z)}$$

Marking the height

$$F(z,u) = \frac{1}{1-zP(1)} \frac{1-\frac{p_1}{p_{-1}}u_1(z)}{1-u\frac{p_1}{p_{-1}}u_1(z)} = \frac{1}{1-p_1 zuE(z)} M_-(z)$$

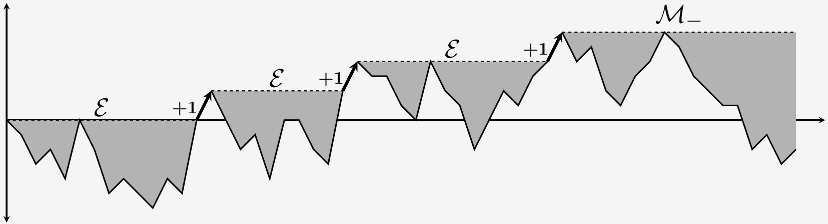

Figure: The first passage decomposition of a Motzkin walks into (negative) excursions and a trailing negative meander.

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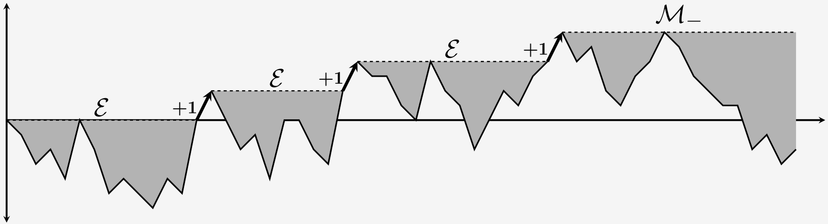
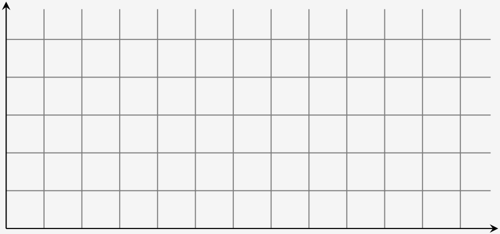


Figure: The first passage decomposition of a Motzkin walks into (negative) excursions and a trailing negative meander.

$$\mathbb{P}[X_n = k] = \frac{[u^k z^n]F(z,u)}{[z^n]F(z,1)} = \frac{[u^k z^n]F(z,u)}{P(1)^n}.$$

Reflection-absorption model

- Lattice: $\mathbb{Z}_+^2$



Absorption model (extends [Banderier–Flajolet, '02])

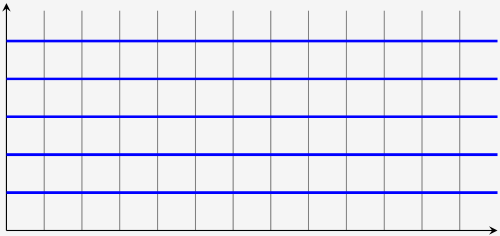
Loss of mass at 0: $P_0(1) < 1$

Reflection model [Banderier–W, '14]

No loss of mass at 0: $P_0(1) = 1$

Reflection-absorption model

- Lattice: $\mathbb{Z}_+^2$
- Altitude $k \neq 0$
  - Weighted step set $\mathcal{S}$
  - $P(u) = \sum_{i=-c}^{d} p_i u^i$



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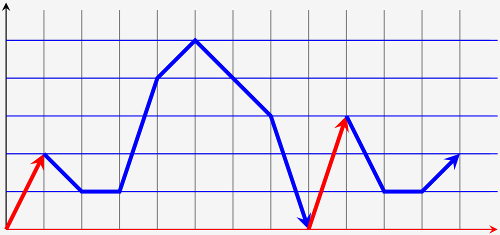
- Altitude $k = 0$
  - Weighted step set $\mathcal{S}_0$
  - $\boldsymbol{P_0(u) = \sum_{i=0}^{d_0} p_{0,i} u^i}$



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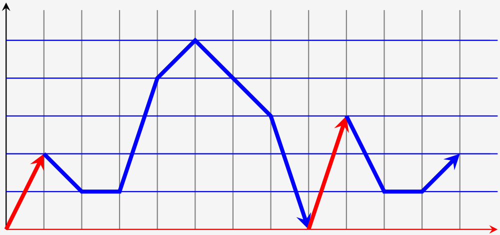

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Final altitude of meanders

Definition
The *final altitude* of a path is defined as the ordinate of its endpoint.

Generating function

$$F(z,u) = \frac{1 - z\left(P(u) - P_0(u)\right)E(z)}{1 - zP(u)}$$

$$E(z) = \frac{(-1)^{c+1}}{zp_{-c}} \prod_{i=1}^{c} u_i(z)$$

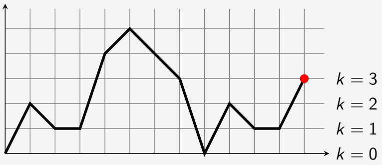

Figure: The final altitude is 3

Meanders of length n and final altitude k

$$\mathbb{P}[X_n = k] = \frac{[z^n u^k]F(z,u)}{[z^n]F(z,1)}$$

Limit laws for the final altitude of meanders

Drift: $\delta = P'(1)$

Theorem (Reflection-absorption model)

	$\delta < 0$	$\delta = 0$		$\delta > 0$
Limit law				
$\mathbb{E}[X_n] \sim$	*const*	$\sqrt{\frac{2}{\pi}}\sqrt{P''(1)n}$	$\sqrt{\frac{\pi}{2}}\sqrt{P''(1)n}$	δn

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Limit laws for the final altitude of meanders

Drift: $\delta = P'(1)$

Theorem (Reflection-absorption model)

	$\delta < 0$	$\delta = 0$		$\delta > 0$
Limit law	**Discrete**			**Gaussian**
$\mathbb{E}[X_n] \sim$	*const*	$\sqrt{\frac{2}{\pi}}\sqrt{P''(1)n}$	$\sqrt{\frac{\pi}{2}}\sqrt{P''(1)n}$	δn

Limit laws for the final altitude of meanders

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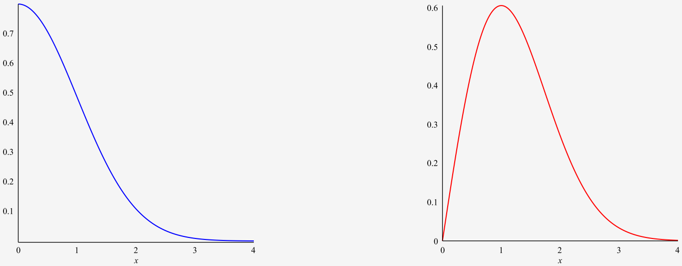


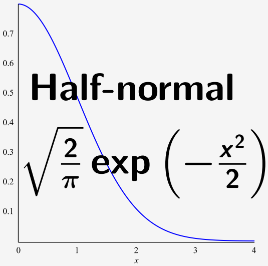
Figure: Limit distributions of final altitude of meanders for drift $\delta = 0$ in the reflection (left) and absorption (right) model.

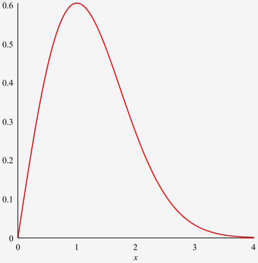
Limit laws for the final altitude of meanders

Drift: $\delta = P'(1)$

Theorem (Reflection-absorption model)

	$\delta < 0$	$\delta = 0$		$\delta > 0$
Limit law	**Discrete**	**Half-normal**		**Gaussian**
$\mathbb{E}[X_n] \sim$	*const*	$\sqrt{\frac{2}{\pi}}\sqrt{P''(1)n}$	$\sqrt{\frac{\pi}{2}}\sqrt{P''(1)n}$	δn





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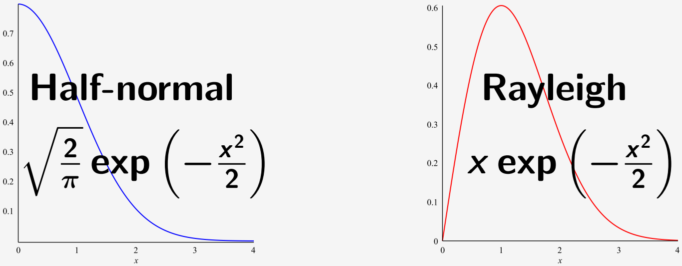
Limit laws for the final altitude of meanders

Drift: $\delta = P'(1)$

Theorem (Reflection-absorption model)

	$\delta < 0$	$\delta = 0$		$\delta > 0$
Limit law	**Discrete**	**Half-normal**	**Rayleigh**	**Gaussian**
$\mathbb{E}[X_n] \sim$	*const*	$\sqrt{\frac{2}{\pi}}\sqrt{P''(1)n}$	$\sqrt{\frac{\pi}{2}}\sqrt{P''(1)n}$	δn



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