Locally Restricted Sequential Structures and Runs of a Subcomposition in Integer Compositions

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Sequential structures

Notation and definition

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- $\mathcal{P}$ denotes a set of *parts* such that for each $n \in \mathbb{N}$ the set $\mathcal{P}_n$ of parts of size n is finite.

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- $\mathrm{SEQ}_k(\mathcal{P})$ denotes the subset of $\mathrm{SEQ}(\mathcal{P})$ consisting of all sequences with exactly k parts. We say that the structures in $\mathrm{SEQ}_k(\mathcal{P})$ have *length* k.

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Example "Composition of salads".

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P_k(x) &= (\log_b e) \sum_{\ell \neq 0} \Gamma(k - 2i\pi\ell \log_b e) \exp(2i\ell\pi x).
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Our approach: Convert runs into *run parts* and study the maximum run part size in a *locally restricted* sequential structure.

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(i) 12 does not occur, and

(ii) no adjacent run parts.

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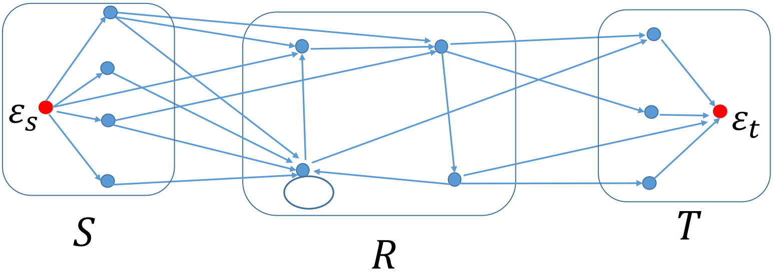
General locally restricted sequential structures can be constructed by adding a block of parts at a time such that the adjacent blocks satisfy the given local restrictions. Such local restrictions on adjacent blocks can be described by a local restriction digraph D such that the vertices are the blocks and arcs indicate legal concatenations.

Local restriction digraph



Start vertices

Recurrent vertices form a strongly connected component.

Terminal vertices

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- There is an arc from a vertex $v \in \mathcal{S} \cup \mathcal{R}$ to a vertex $u \in \mathcal{R}$ iff the concatenation vu is alternating. Arcs from $\mathcal{R}$ to $\mathcal{T} \cup \mathcal{R}$ are defined similarly.

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Let $\mathcal{W}$ denote the set of all directed walks in D from ε_s to ε_t, and let $\mathrm{SEQ}(\mathcal{P}; D)$ denote the class of all structures of the concatenation form $v_1 v_2 \cdots v_j$, where $\varepsilon_s v_1 v_2 \cdots v_j \varepsilon_t \in \mathcal{W}$.

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We say that $\mathrm{SEQ}(\mathcal{P}; D)$ is a locally restricted class of sequential structures associated with D.
We can apply the *transfer matrix method* to enumerate $\mathrm{SEQ}(\mathcal{P}; D)$.
When there are finitely many parts, the digraph D is finite and the corresponding transfer matrix is finite. We can apply Perron-Frobenius Theorem.

The transfer matrix

In general, let $r_1, r_2, \ldots$ be an ordered list of vertices in $\mathcal{R}$. We define the transfer matrix $T(z)$ such that the (i, j)th entry of $T(z)$ is $T_{i,j}(z) = z^{|r_i|+|r_j|}$ if there is an arc in $D_{\mathcal{R}}$ from r_i to r_j; otherwise $T_{i,j}(z) = 0$.

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Define the *weight* of an arc (v, w) to be $z^{|v|+|w|}$, and the weight of a directed walk to be the product of the weights of all the arcs in the walk.

Define the *start vector* $\mathbf{s}(z)$ and *terminal vector* $\mathbf{t}(z)$ as follows. The ith component of $\mathbf{s}(z)$ is the sum of weights of all directed walks from ε_s to r_i, and the jth component of $\mathbf{t}(z)$ is the sum of weights of all directed walks from r_j to ε_t.

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In general, let $r_1, r_2, \ldots$ be an ordered list of vertices in $\mathcal{R}$. We define the transfer matrix $T(z)$ such that the (i,j)th entry of $T(z)$ is $T_{i,j}(z) = z^{|r_i|+|r_j|}$ if there is an arc in $D_{\mathcal{R}}$ from r_i to r_j; otherwise $T_{i,j}(z) = 0$.

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Then we have

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Regular local restrictions

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- $\lambda'(z) > 0$.

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Consequently (Bender-Canfield, 09)

- $\lambda(z)$ is analytic in a neighbor of $(0, \rho)$, and $A(z) = \frac{g(z)}{1-\lambda(z)} + h(z)$ for some functions $g(z)$ and $h(z)$ which are analytic in $|z| \leq \rho$.

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Let $\zeta_n(k)$ be the number of occurrences of part k in $\mathbf{a}$, M_n be the maximum part size in $\mathbf{a}$, and $g_n(k)$ be the probability that $\mathbf{a}$ contains exactly k parts of size M_n. Then (all logs are to the base $1/r$)

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NEW: The above results are now extended to general sequential structures and *free parts*.

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The values of r and C

(a) All compositions for any given composition $\mathbf{c}$, where $r = 1/2$ and $C = \frac{1}{2}(1 - 2^{-|\mathbf{c}|})^2$. In particular, $C = 1/8$ when $\mathbf{c} = 1$, which gives Gafni's result.

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(c) Alternating compositions, where $r \doteq 0.636282$, and for $\mathbf{c} = 12$ or $\mathbf{c} = 21$, $C \doteq 0.062020$.

Thank you !

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