The Register Function and Reductions of Binary Trees and Lattice Paths

Benjamin Hackl

joint work with
Clemens Heuberger and *Helmut Prodinger*

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mso DK modeling simulation optimization

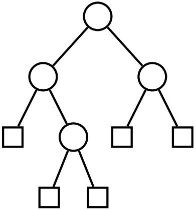
KARL POPPER KOLLEG

FWF Der Wissenschaftsfonds.

Trimming binary trees

Binary trees can be "trimmed" by the following strategy:

- Remove all leaves
- Merge nodes with only one descendant



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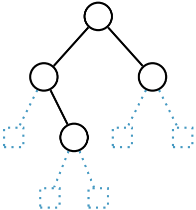

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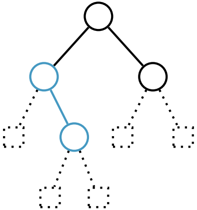

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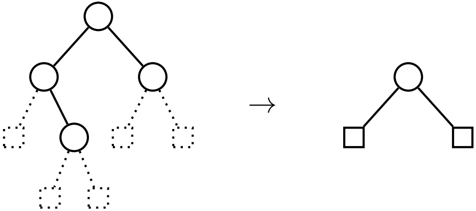

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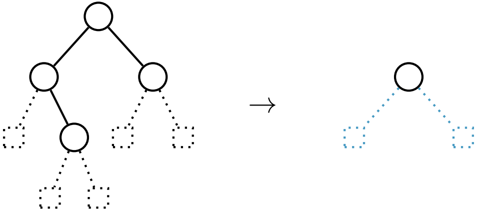

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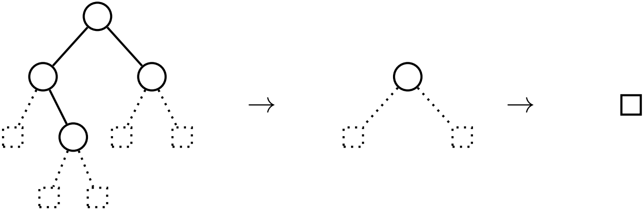

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Label all nodes in the tree by the following rules:

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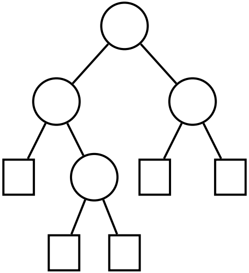
- Otherwise: take the maximum



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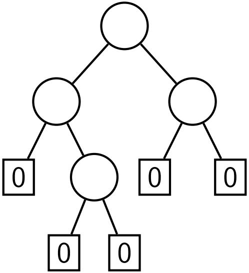

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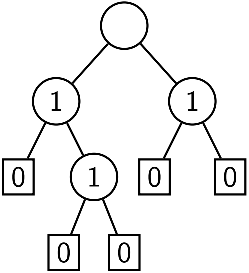

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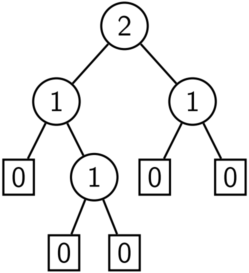

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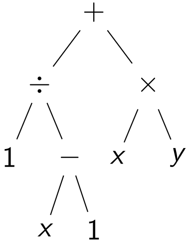

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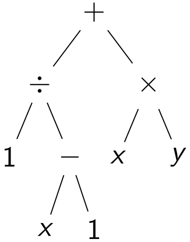
  - Branching complexity of river networks (e.g. Danube: 9)



The register function – selection of known results

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- Louchard, Prodinger (2008): register function for directed lattice paths

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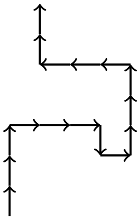
Reduction of a simple, two-dimensional lattice path (i.e. a sequence of $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$):



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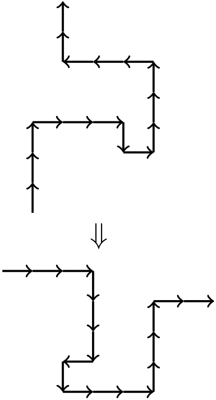

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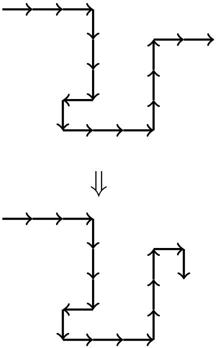

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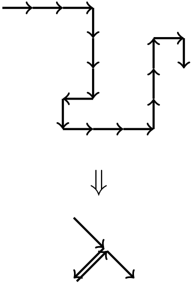

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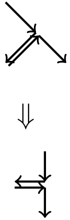

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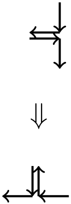

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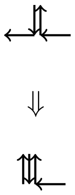

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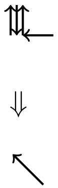

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$\Downarrow$

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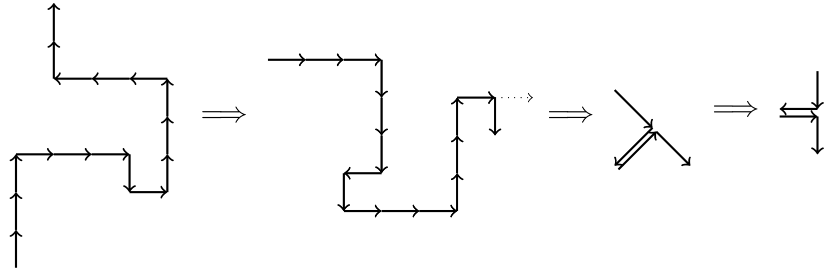
Reduction – Example



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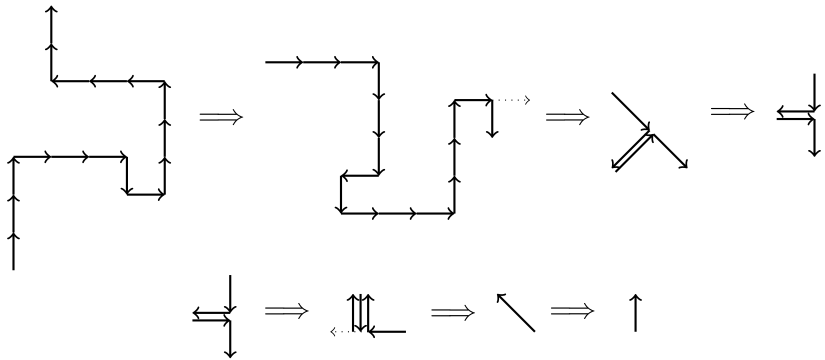

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The generating function of simple two-dimensional lattice paths of length ≥ 1, $L(z) = \frac{4z}{1-4z}$, fulfills the functional equation

$$L(z) = 4z + 4L\left(\frac{z^2}{(1-2z)^2}\right).$$

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Can be checked directly—or proven combinatorially!

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Functional equation (combinatorial proof)

Read the reduction *backwards*:

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Adding $4z$ (for $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$) then counts all paths. $\square$

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- Overall:

$$L_r^{=}(z) = 4^{r+1} \frac{u}{(1+u)^2}\Big|_{u \mapsto u^{2^r}} = 4^{r+1} \frac{u^{2^r}}{(1+u^{2^r})^2}$$

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Compactification degree – Random variables

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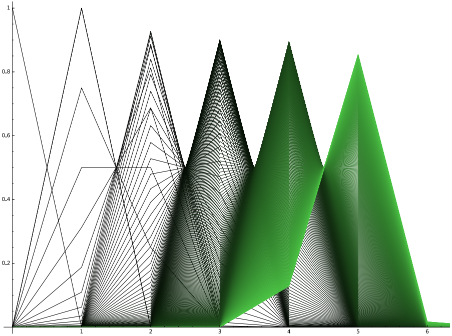
- Probability densities of X_1 up to X_{512}:



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- With $z = \frac{u}{(1+u)^2}$ and $u = e^{-t}$, we have

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$\rightsquigarrow$ Local expansion for $t \to 0$ ($z \to \frac{1}{4}$) via Mellin transform

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- Mellin inversion:

$$G(z) = \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} \Gamma(s)\zeta(s-1)\frac{2^{2-s}}{1-2^{2-s}} t^{-s}\, ds$$

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- By basic properties of the Mellin transform we find

$$G^*(s) = \Gamma(s)\zeta(s-1)\frac{2^{2-s}}{1-2^{2-s}}$$

- Double pole at $s = 2$, simple poles at $s = 2 + \frac{2\pi i}{\log 2}k = 2 + \chi_k$ for $k \in \mathbb{Z} \setminus \{0\}$
- Mellin inversion:

$$G(z) = \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} \Gamma(s)\zeta(s-1)\frac{2^{2-s}}{1-2^{2-s}}t^{-s}\, ds$$

- Obtain contribution by shifting line of integration

Analysis of $\mathbb{E}X_n$ (3)

- Residue at $s = 2$:

$$-\frac{4}{\log 2} t^{-2} \log t + \left(\frac{4}{\log 2} - 2\right) t^{-2}$$

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- Substituting z back for t and expanding locally for $z \to \frac{1}{4}$ yields

$$-\frac{\log(1-4z)}{\log 2\,(1-4z)} + \frac{2 - 3\log 2}{\log 2\,(1-4z)} + \frac{\log 2 - 1}{\log 2} + \frac{\log(1-4z)}{3\log 2} + O(1-4z)$$

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Analysis of $\mathbb{E}X_n$ (4)

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$$\log_4 n + \frac{\gamma + 2 - 3\log 2}{2\log 2} + O(n^{-2}).$$

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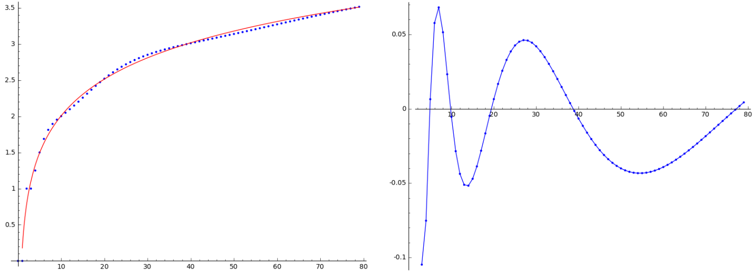
- Plot against exact values (left: comparison, right: difference):



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Analysis of $\mathbb{E}X_n$ (5)

Collecting the contributions at $s = 2 + \chi_k$ yields:

Theorem (H.–Heuberger–Prodinger, 2016)

The expected compactification degree among all simple 2D lattice paths of length n admits the asymptotic expansion

$$\mathbb{E}X_n = \log_4 n + \frac{\gamma + 2 - 3\log 2}{2\log 2} + \delta_1(\log_4 n) + O(n^{-1}),$$

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where

$$\delta_1(x) = \frac{1}{\log 2} \sum_{k \neq 0} \frac{\Gamma(2+\chi_k)\zeta(1+\chi_k)}{\Gamma(1+\chi_k/2)} e^{2k\pi i x}$$

is a small 1-periodic fluctuation.

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Analysis of $\mathbb{V}X_n$

Similarly: variance $\mathbb{V}X_n$ can be determined.

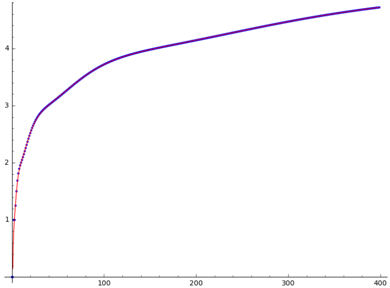
Theorem (H.–Heuberger–Prodinger, 2016)

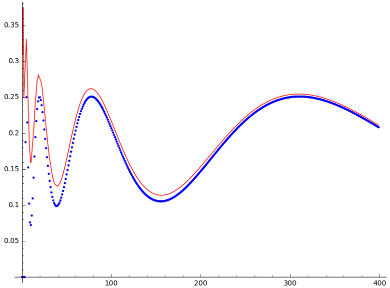
The corresponding variance is given by

$$\mathbb{V}X_n = \frac{\pi^2 - 24\log^2\pi - 48\zeta''(0) - 24}{24\log^2 2} - \frac{2\log\pi}{\log 2} - \frac{11}{12} + \delta_2(\log_4 n) + \frac{\gamma + 2 - 3\log 2}{\log 2}\delta_1(\log_4 n) + \delta_1^2(\log_4 n) + O\Big(\frac{1}{\log n}\Big),$$

where $\delta_1(x)$ is defined as above and $\delta_2(x)$ is a small 1-periodic fluctuation as well.

Expectation and Variance: exact vs. asymptotic





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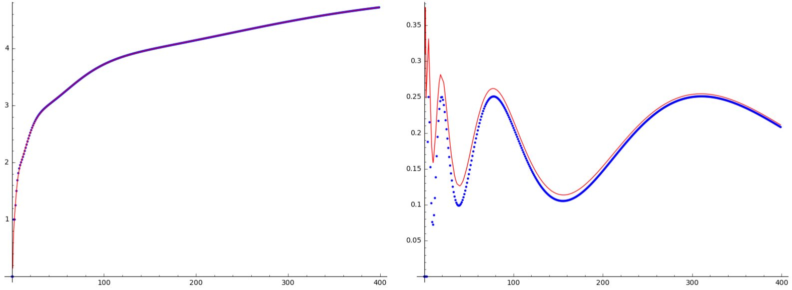

Computations $\rightsquigarrow$ Asymptotic Expansions in SageMath!

Fringe Analysis

- *Fringe*: lattice path together with all reductions

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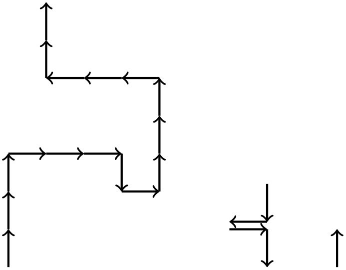

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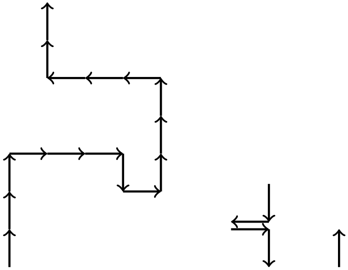

- How large is the rth fringe and the entire fringe on average?

Bivariate generating function

- $H_r(z, v)\dots$ BGF counting path length (with z) and rth fringe size (with v)

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$$H_0(z,v) = \frac{4zv}{1-4zv}, \quad H_r(z,v) = 4H_{r-1}\left(\left(\frac{z}{1-2z}\right)^2, v\right)$$

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- Explicit solution with $z = \frac{u}{(1+u)^2}$:

$$H_r(z, v) = \frac{4^{r+1} u^{2^r} v}{(1 + u^{2^r})^2 - 4u^{2^r} v}$$

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Size of rth fringe

Theorem (H.–Heuberger–Prodinger, 2016)

The expectation $E^L_{n;r}$ and variance $V^L_{n;r}$ of the rth fringe size of a random path of length n have the asymptotic expansions

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Size of rth fringe

Theorem (H.–Heuberger–Prodinger, 2016)

The expectation $E_{n;r}^L$ and variance $V_{n;r}^L$ of the rth fringe size of a random path of length n have the asymptotic expansions

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where $\theta_r = \frac{4}{2+2\cos(2\pi/2^r)} > 1$.
For $r > 0$, the random variables modeling the rth fringe size of lattice paths of length n are asymptotically normally distributed.

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Overall fringe size

Strategy: sum over $H_r(z, v)$, expansion via Mellin transform, singularity analysis.

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Theorem (H.–Heuberger–Prodinger, 2016)

The expected fringe size E_n^L for a random path of length n admits the asymptotic expansion

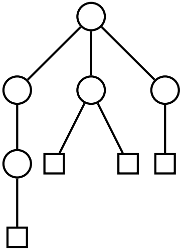
$$E_n^L = \frac{4}{3}n + \frac{1}{3}\log_4 n + \frac{5 + 3\gamma - 11\log 2}{18\log 2} + \delta(\log_4 n) + O(n^{-1}\log n),$$

where $\delta(x)$ is a 1-periodic fluctuation of mean zero with

$$\delta(x) = \frac{2}{3\sqrt{\pi}\log 2}\sum_{k\neq 0}\Gamma\Big(\frac{3+\chi_k}{2}\Big)\big(2\zeta(\chi_k - 1) + \zeta(\chi_k + 1)\big)e^{2k\pi ix}.$$

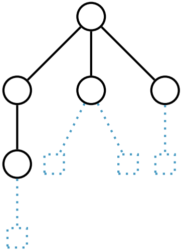
Outlook

$\rightsquigarrow$ reductions of rooted plane trees



Outlook

⇝ reductions of rooted plane trees



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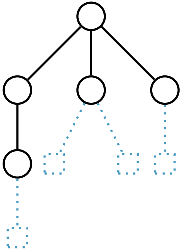

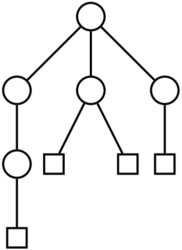

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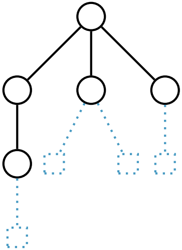

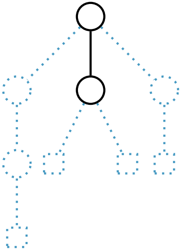

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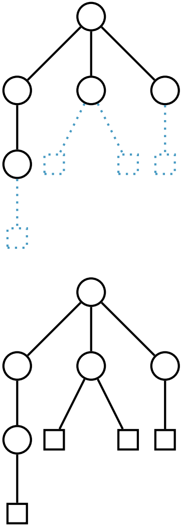

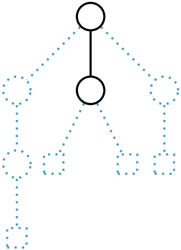

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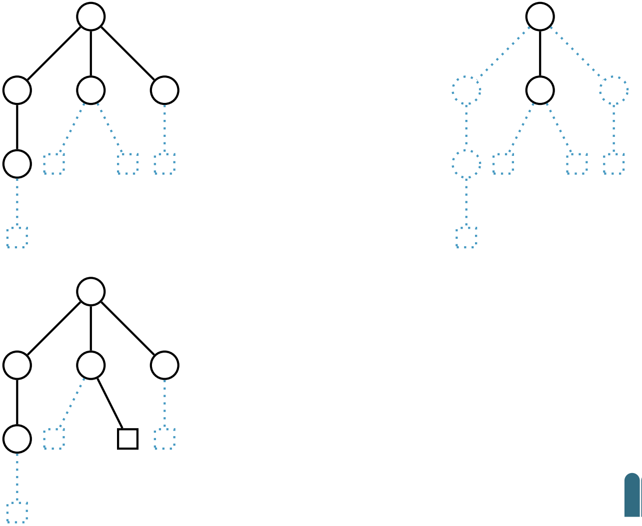

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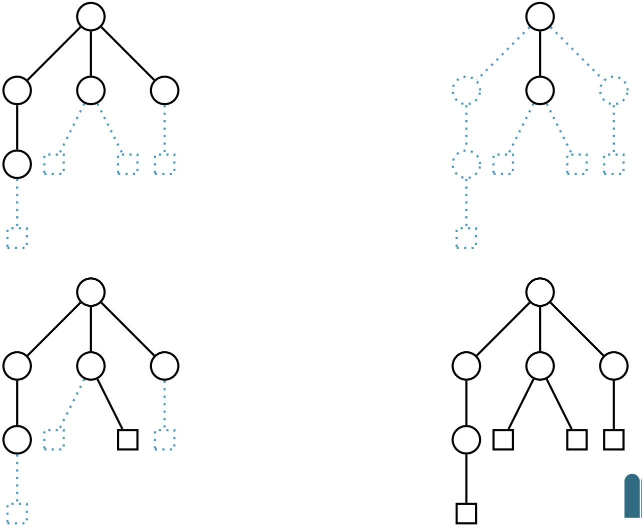

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