Recurrences for generating polynomials

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(joint work Amanda Lohss)

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Some recently encountered recurrences for polynomials

In what follows $(P_n(x))$, $Q_n(x))$, $(A_n(x)), \cdots$ are sequences of polynomials.

- **Laborde Zubieta** (and also **H. & Lohss (2015)**):

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\begin{aligned}
P_n'(x) &= nP_{n-1}(x) + 2(1-x)P_{n-1}'(x), \\
P_0(x) &= 1.
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- **Acan, H. (2016)**:

$$\begin{aligned} A_n(x) &= (2n-1)A_{n-1}(x) + x(x-1)A_{n-1}'(x), \\ A_0(x) &= x. \end{aligned}$$

More examples

- **H., Janson (2014)**: for $a, b > 0$ (but could be complex)

$$\begin{aligned} P_{n,a,b}(x) &= ((n-1+b)x+a)P_{n-1,a,b}(x) + x(1-x)P'_{n-1,a,b}(x) \\ P_{0,a,b}(x) &= 1. \end{aligned}$$

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- These generalize classical Eulerian polynomials. That is:

$$P_{n,1,0}(x) = E_n(x) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k,$$

where $\left\langle {n \atop k} \right\rangle$ is the number of permutations of $[n]$ with exactly k ascents and the recurrence is:

$$E_n(x) = ((n-1)x+1)E_{n-1}(x) + x(1-x)E'_{n-1}(x).$$

Last examples

- **Dasse–Hartaut, H. (2013)** (implicit):

$$\begin{aligned} V_n(x) &= ((2n-1)x+1)V_{n-1}(x) + 2x(1-x)V'_{n-1}(x) \\ V_0(x) &= 1 \end{aligned}$$

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- **Aval, Boussicault, Nadeau (2011)**:

$$\begin{aligned} B_n(x) &= nx(x+1)B_{n-1}(x) + x(1-x^2)B'_{n-1}(x), \\ B_0(x) &= x. \end{aligned}$$

General Situation

We are given a sequence of polynomials

$$P_n(x) = \sum_{k=0}^{m} p_{n,k} x^k, \quad n \geq 0$$

that satisfy a recurrence

$$\left.\begin{matrix} P_n'(x) \\ \text{or} \\ P_n(x) \end{matrix}\right\} = f_n(x) P_{n-1}(x) + g_n(x) P_{n-1}'(x)$$

for some sequences of (low–degree) polynomials (f_n), (g_n) and for some reason

$$g_n(1) = 0.$$

Probabilistic Motivation

If

$$P_n(x) = \sum_{k\geq 0} p_{n,k} x^x, \quad p_{n,k} \geq 0,$$

then

$$\frac{P_n(x)}{P_n(1)} = \sum_{k\geq 0} \frac{p_{n,k}}{P_n(1)} x^k = \mathbb{E} x^{X_n},$$

where, for $n \geq 1$, X_n is a random variable such that

$$\mathbb{P}(X_n = k) = \frac{p_{n,k}}{P_n(1)} = \frac{p_{n,k}}{\sum_j p_{n,j}}, \quad k \geq 0.$$

Method of Moments

Suppose X is a random variable *uniquely determined* by its sequence of moments (not all are!)

$$\mu_r := \mathbb{E}X^r, \quad r = 1, 2, \dots$$

Let (X_n) be a sequence of random variables such that

$$(*) \qquad \mathbb{E}X_n^r \longrightarrow \mathbb{E}X^r, \quad r = 1, 2 \dots$$

Then

$$X_n \xrightarrow{d} X, \quad \text{as } n \to \infty.$$

Factorial moments version: the same conclusion holds if $(*)$ is replaced by the convergence of factorial moments, i.e.

$$\mathbb{E}(X_n)_r \longrightarrow \mathbb{E}(X)_r, \quad r = 1, 2 \dots$$

where

$$\mathbb{E}(X)_r := \mathbb{E}X(X-1)\dots(X-(r-1)).$$

Final Connection

If X is a random variable with probability generating function

$$\phi(x) := \mathbb{E}x^X,$$

then for $r = 1, 2, \ldots$

$$\mathbb{E}(X)_r = \phi^{(r)}(1).$$

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In our case, this would mean

$$\mathbb{E}(X_n)_r = \frac{P_n^{(r)}(1)}{P_n(1)}$$

and thus we are interested in computing $P_n^{(r)}(1)$.

For the recurrence

$$P_n(x) = f_n(x)P_{n-1}(x) + g_n(x)P'_{n-1}(x)$$

Leibniz formula gives

$$\begin{aligned} P_n^{(r)}(x) &= (f_n(x)P_{n-1}(x))^{(r)} + (g_n(x)P'_{n-1}(x))^{(r)} \\ &= \sum_{k=0}^{r} \binom{r}{k} f_n^{(k)}(x)P_{n-1}^{(r-k)}(x) \\ &+ \sum_{k=0}^{r} \binom{r}{k} g_n^{(k)}(x)P_{n-1}^{(r+1-k)}(x). \end{aligned}$$

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So, if f_n and g_n are low–degree polynomials then one obtains reasonably simple recurrence for $P_n^{(r)}(1)$ that perhaps can be worked with.

Example: Laborde Zubieta recurrence

Proposition (H. & Lohss)

Let

$$P_n'(x) = f_n P_{n-1}(x) + (x-1) g_n P_{n-1}'(x)$$

where f_n and g_n are sequences of constants such that

$$\frac{g_n}{f_n} \to 0 \quad \text{and} \quad f_n \frac{P_{n-1}(1)}{P_n(1)} \to \lambda > 0, \; n \to \infty.$$

Then, for the corresponding random variables (X_n),

$$X_n \xrightarrow{d} Pois(\lambda),$$

where $Pois(\lambda)$ is a Poisson random variable with parameter λ.

Proof of Proposition

Recall that if X is $Pois(\lambda)$ then

$$\mathbb{E}(X)_r \;=\; \lambda^r.$$

Since $f_n(x) = f_n$ and $g_n(x) = (1-x)g_n$ are of degree 0 and 1, $g_n(1) = 0$ and $g_n'(1) = -g_n$, Leibniz differentiation

$$P_n^{(r)}(x) \;=\; \sum_{k=0}^{r-1} \binom{r-1}{k} f_n^{(k)}(x) P_{n-1}^{(r-1-k)}(x) + \sum_{k=0}^{r-1} \binom{r-1}{k} g_n^{(k)}(x) P_{n-1}^{(r-k)}(x).$$

gives

$$P_n^{(r)}(1) \;=\; \left(f_n - (r-1)g_n\right) P_{n-1}^{(r-1)}(1).$$

Normalize and iterate...

Hence

$$\frac{P_n^{(r)}(1)}{P_n(1)} = \left((f_n - (r-1)g_n) \frac{P_{n-1}(1)}{P_n(1)} \right) \frac{P_{n-1}^{(r-1)}(1)}{P_{n-1}(1)}.$$

Hence

$$\frac{P_n^{(r)}(1)}{P_n(1)} = \left((f_n - (r-1)g_n)\frac{P_{n-1}(1)}{P_n(1)}\right)\frac{P_{n-1}^{(r-1)}(1)}{P_{n-1}(1)}.$$

Iterating $r-1$ more times gives

$$\begin{aligned}\frac{P_n^{(r)}(1)}{P_n(1)} &= \left(\prod_{j=0}^{r-1}(f_{n-j} - (r-1)g_{n-j})\frac{P_{n-j-1}(1)}{P_{n-j}(1)}\right)\overbrace{\frac{P_{n-r}^{(r-r)}(1)}{P_{n-r}(1)}}^{=1} \\ &= \prod_{j=0}^{r-1}\left(f_{n-j}\frac{P_{n-j-1}(1)}{P_{n-j}(1)}\left(1-(r-1)\frac{g_{n-j}}{f_{n-j}}\right)\right) \to \lambda^r\end{aligned}$$

b/c by our assumptions, for every fixed $j \geq 0$, the first term under the product goes to λ and the second to 1.

Real–rootedness and convergence to normal

If all roots of P_n are real then

$$P_n(x) = p_{n,m}\prod_{j=1}^{m}(x+q_{n,j}), \quad q_{n,1},\dots,q_{n,m} > 0.$$

Real–rootedness and convergence to normal

If all roots of P_n are real then

$$P_n(x) = p_{n,m}\prod_{j=1}^{m}(x+q_{n,j}), \quad q_{n,1},\dots,q_{n,m}>0.$$

Thus,

$$\mathbb{E}x^{X_n} = \frac{P_n(x)}{P_n(1)} = \prod_{k=1}^{m}\frac{x+q_{n,k}}{1+q_{n,k}} = \prod_{k=1}^{m}\left(x\cdot\frac{1}{1+q_{n,k}} + 1\cdot\frac{q_{n,k}}{1+q_{n,k}}\right).$$

The factor on the right is the probability generating function of a random variable $\xi_{n,k}$ which takes only values 0 or 1 and the product says that they are (stochastically) independent. So

$$X_n = \sum_{k=1}^{m}\xi_{n,k}.$$

Since

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$$var(X_n) \quad = \quad D_n + E_n - E_n^2,$$

where $E_n := \mathbb{E}X_n$ and $D_n := \mathbb{E}(X_n)_2$.

The variance

So, the variance is expressible in terms of P_n and its first two derivatives evaluated at 1. In fact, suppressing evaluations at 1,

$$E_n = \frac{P_n'}{P_n} = \left(1 + \frac{g_n'}{f_n}\right) E_{n-1} + \frac{f_n'}{f_n}.$$

and

$$D_n = \frac{P_n''}{P_n} = \left(1 + \frac{2g_n'}{f_n}\right) D_{n-1} + \frac{2f_n' + g_n''}{f_n} E_{n-1} + \frac{f_n''}{f_n}$$

and it is typically easy to see from here whether $var(X_n) \to \infty$ or not.

Conditions for the real–rootedness

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Conditions for the real–rootedness

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- L. L. Liu and Y. Wang A unified approach to polynomial sequences with only real zeros. *Adv. Appl. Math.*, 2007 consider

$$P_n(x) = f_n(x)P_{n-1}(x) + g_n(x)P'_{n-1}(x) + h_n(x)P_{n-2}(x),$$

under extra assumptions (but not on the degrees f_n, g_n, and h_n). In any case, the minimum assumption is that

$$g_n(x) \le 0, \quad h_n(x) \le 0 \quad \text{for} \quad x \le 0.$$

- D. Dominici, K. Driver, K Jordaan, Polynomial solutions of differential–difference equations. *J. Approx. Theory*, 2011 consider

$$(*) \qquad P_n(x) = f_n(x)P_{n-1}(x) + g_n(x)P'_{n-1}(x),$$

where f_n's have degrees at most 1 and g_n's at most 2. Using integrating factor trick, they write $(*)$ as

$$P_n(x) = \frac{g_n(x)}{K_n(x)}\left[K_n(x)P_{n-1}(x)\right]', \; K_n(x) = \exp\int_0^x \frac{f_n(t)}{g_n(t)}dt$$

and rely on Rolle's theorem (applied to $K_n(x)P_{n-1}(x)$).

Asymptotic normality in few cases (V_n, E_n are linear in n)

This unifies proofs of several asymptotic normality results.

- **H., Janson (2014)**: (mimicking **Frobenius (1910)** proof of Eulerian case)

$$P_{n,a,b}(x) \quad = \quad ((n-1+b)x+a)P_{n-1,a,b}(x)+x(1-x)P'_{n-1,a,b}(x).$$

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NOTE: The recurrence goes back to at least **McMahon (1920)**, but the closed form of the bivariate generating function was not found till **Frassens (2006)**.

Left out piece

Aval, Boussicault, Nadeau (2011) recurrence

$$B_n(x) \quad = \quad nx(x+1)B_{n-1}(x) + x(1-x^2)B'_{n-1}(x)$$

does not fall in either of the cases:

- $nx(1-x^2) \le 0$ fails for $x \le 0$ and
- Both $x(1+x)$ and $x(1-x^2)$ have too high degrees for **Dominici, Driver, Jordaan** to apply.

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- Both $x(1+x)$ and $x(1-x^2)$ have too high degrees for **Dominici, Driver, Jordaan** to apply.
- But their method can be adapted since all B_n's have common roots at -1 and 0 (**H. & Lohss**).
- Integrating factor is $K_n(x) = (1-x)^{-n}$ and the $(*)$ version of the recurrence becomes

$$B_n(x) = x(1+x)(1-x)^{n+1}\left[K_n(x)B_{n-1}(x)\right]', \quad n \ge 1$$
$$B_0(x) = x.$$

Left out piece

It is very easy to show by induction, that for the recurrence

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with $K_n(x) = (1-x)^{-n}$, we have

- $B_n(x)$ has degree $n+1$, for $n \geq 0$ and
- for $n \geq 1$ the roots of $B_n(x)$ are simple and are in the interval $[-1, 0]$. In fact, the roots (other than -1 and 0) interlace with the roots of $B_{n+1}(x)$.

Super left out piece

This leaves us with the case:

- **Acan, H. (2016)**:

$$A_n(x) \quad = \quad (2n-1)A_{n-1}(x) + x(x-1)A'_{n-1}(x),$$

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$$\frac{X_n}{2\sqrt{n}} \xrightarrow{d} X, \quad X \text{ has density } 2xe^{-x^2} \text{ for } x > 0.$$

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- In fact, in this case $\mathbb{E}X_n \sim 2\sqrt{n}$, $var(X_n) \sim (4-\pi)n$.
- We are lacking generality that would let us classify the recurrences (even that simple) according to the limit law they lead to and I think it would be interesting to develop that.

Tree–Like Tableaux (Aval, Boussicault, Nadeau (2011))

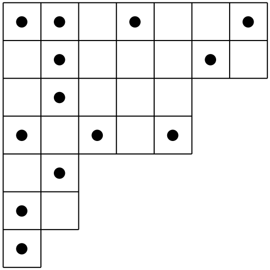


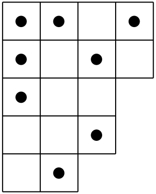
Figure: A tree–like tableaux of size 13.

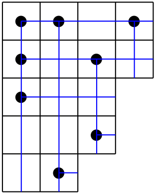
Definition

A tree-like tableaux of size n is a Ferrers diagrams of half-perimeter $n+1$ such that,

1. The box in the first column and first row is pointed.
2. Either all boxes to the left of a pointed box is empty or all boxes above are empty.
3. Every row and every column contains at least one point.

Motivation





- Natural tree structure
- Connection to the PASEP

Symmetric Tree–like Tableaux

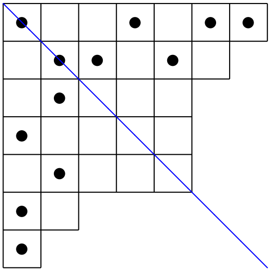


Figure: A symmetric tree–like tableaux of size 13.

Definition

A symmetric tree–like tableau is a tree–like tableau that is symmetric with respect the main diagonal

- It necessarily is of an odd size.
- Out of all $(2n+1)!$ tree–like tableaux of size $2n+1$, $2^n n!$ are symmetric.

Occupied Corners

Laborde Zubieta recurrence is for the generating polynomials of the number of occupied corners in tree–like tableaux of size n. He proved that the expected value is 1 and that the variance is $(n-2)/n$.
We extended that to:

Theorem (H., & Lohss)

As $n \to \infty$,

- *the limiting distribution of the number of occupied corners in a random tree–like tableau of size n is* $\mathrm{Pois}(1)$.
- *the limiting distribution of the number of occupied corners in a random symmetric tree–like tableau of size $2n+1$ is* $2 \times \mathrm{Pois}(1/2)$.

Diagonal boxes in symmetric tableaux

Similarly, **ABN** recurrence is for the generating polynomials of the number of diagonal boxes in symmetric tree–like tableaux of size $2n+1$. They proved that the expected value is $3(n+1)/4$.
We extended that to:

Theorem (H., & Lohss)

If D_n is the number of diagonal boxes in a random symmetric tree–like tableau of size then:

- $var(D_n) = \dfrac{7(n+1)}{48}$.
- *As* $n \to \infty$,

$$\frac{D_n - 3n/4}{\sqrt{7n/48}} \xrightarrow{d} N(0,1),$$

where $N(0,1)$ *is the standard normal random variable.*

Thank you :) :-)