Corners in Tree–Like Tableaux

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- Definition and Motivation
- Two Conjectures
- Corners in Tree–Like Tableaux
- Corners in Symmetric Tree–Like Tableaux

Tree–Like Tableaux (Aval, Boussicault, Nadeau 2011)

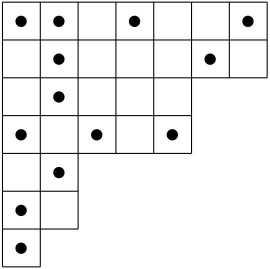


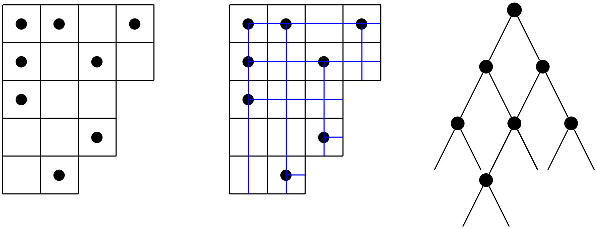
Figure: A tree–like tableaux of size 13. There are $n!$ tableaux of size n.

Definition

A tree-like tableaux of size n is a Ferrers diagrams of half-perimeter $n+1$ such that,

1. The box in the first column and first row is pointed.
2. Either all boxes to the left of a pointed box is empty or all boxes above are empty.
3. Every row and every column contains at least one point.

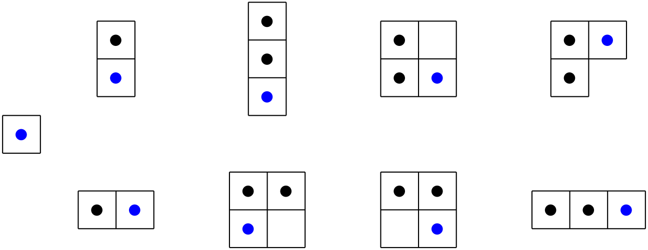
Natural Tree Structure



- Natural tree structure
- Provides combinatorical formula for the equilibrium state of the PASEP

Insertion Procedure

- Definition: The "special point" is the right-most point among those that occur at the bottom of a column.
- Add a pointed column for each north step and a pointed row for each west step.
- If the step is below the special point, add a ribbon.



Main Objective: Laborde Zubieta 2015

Conjecture: The number of corners in tree–like tableaux of size n is $n! \times \frac{n+4}{6}$.

Conjecture: The number of corners in symmetric tree–like tableaux of size $2n+1$ is $2^n \times n! \times \frac{4n+13}{12}$.

Permutation Tableaux

0	1	0	0	1	1
0	0	1	1		
0	1	1	1		
0					
1					

Figure: A permutation tableaux of size 12.

Definition

A permutation tableaux of size n is a Ferrers diagram of half-perimeter n such that

1. There is at least one 1 in every column.
2. There is no 0 with a 1 above it and a 1 to the left of it simultaneously.

The Bijection: (Aval, Boussicault, Nadeau 2011)

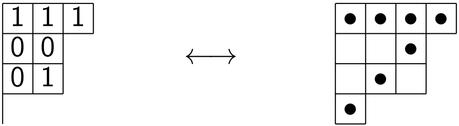


Figure: An example of the bijection

$$c(\mathcal{T}_n) = c(\mathcal{P}_n) + |\{P \in \mathcal{P}_n : \ M_n(P) = S\}| = c(\mathcal{P}_n) + (n-1)!$$

Theorem (Hitczenko, L.)

For permutation tableaux of size n,

$$\mathbb{E}_n C_n = \frac{n+4}{6} - \frac{1}{n}.$$

Symmetric Tree–Like Tableaux

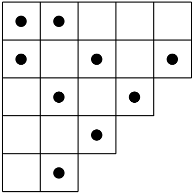


Figure: A symmetric tree–like tableaux of size 9. There are $2^n \cdot n!$ tableaux of size n.

Type-B Permutation Tableaux

1		
0	0	
0	1	1
0	1	
0	0	

Figure: A type-B permutation tableaux of size 6.

Definition

A type-B permutation tableaux of size n is a shifted Ferrers diagram of half-perimeter n such that,

1. The rules of permutation tableaux are satisfied.
2. If there is a 0 on the diagonal, it is a 0-row.

The Bijection: (Aval, Boussicault, Nadeau 2011)

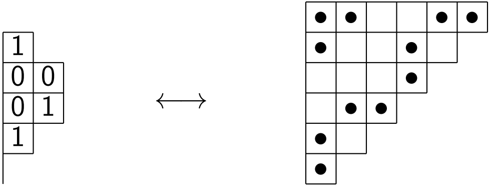


Figure: An example of the bijection

1. Add a column and a root point then point unrestricted rows.
2. Replace all 0_R's with points (except on 0-rows).
3. Replace all non-diagonal 1_T's with points.
4. Make symmetric.

The Bijection

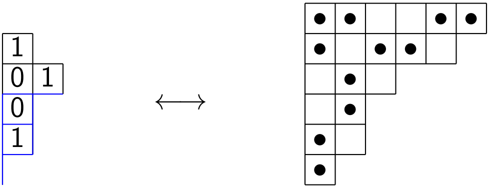


Figure: Transformation of the Shape

Key Relationship:

$$\begin{aligned} c(\mathcal{T}^{sym}_{2n+1}) &= 2c(\mathcal{B}_n) + 2|\{B \in \mathcal{B}_n : \ M_n(B) = S\}| + |\{B \in \mathcal{B}_n : \ M_1(B) = W\}| \\ &= c(\mathcal{B}_n) + 2^n(n-1)! + 2^{n-1}n!, \end{aligned}$$

Relationship Between Measures

If U_{n-1} is the number of unrestricted rows, then there are $2^{U_{n-1}+1}$ extensions. Therefore, for all $B \in \mathcal{B}_{n-1}$,

$$\mathbb{P}_n(B) = \frac{2^{U_{n-1}(B)+1}}{|\mathcal{B}_n|} = 2^{U_{n-1}(B)+1} \frac{|\mathcal{B}_{n-1}|}{|\mathcal{B}_n|} \mathbb{P}_{n-1}(B)$$

Thus, for any random variable X on $\mathcal{B}_{n-1}$,

$$\mathbb{E}_n(X) = \frac{2|\mathcal{B}_{n-1}|}{|\mathcal{B}_n|} \mathbb{E}_{n-1}(2^{U_{n-1}(B_{n-1})} X)$$

The Random Variable U_n

$$\begin{aligned}
\mathbb{P}(U_n = U_{n-1} + 1 | F_{n-1}) &= \frac{2}{2^{U_{n-1}+1}} = \frac{1}{2^{U_{n-1}}}. \\
\mathbb{P}(U_n = k | F_{n-1}) &= \frac{1}{2^{U_{n-1}+1}} \left(\binom{U_{n-1}}{k-1} + \binom{U_{n-1}}{k-1} \right) \\
&= \frac{1}{2^{U_{n-1}}} \binom{U_{n-1}}{k-1}.
\end{aligned}$$

Therefore,

$$\mathcal{L}(U_n | \mathcal{F}_{n-1}) = 1 + \mathrm{Bin}(U_{n-1}, 1/2).$$

Simple Illustration

Proposition (known but new proof: Hitczenko, L.)

For all $n \geq 0$, $|\mathcal{B}_n| = 2^n n!$.

$$|\mathcal{B}_n| = \sum_{B \in \mathcal{B}_{n-1}} 2^{U_{n-1}(B)+1}$$

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$$
\begin{aligned}
|\mathcal{B}_n| &= \sum_{B \in \mathcal{B}_{n-1}} 2^{U_{n-1}(B)+1} \\
&= |\mathcal{B}_{n-1}| \mathbb{E}_{n-1} \left(2^{U_{n-1}+1} \right) \\
&= 2|\mathcal{B}_{n-1}| \mathbb{E}_{n-1} \mathbb{E} \left(2^{U_{n-1}} | U_{n-2} \right) \\
&= 2|\mathcal{B}_{n-1}| \mathbb{E}_{n-1} \mathbb{E} \left(2^{1+\mathrm{Bin}(U_{n-2})} | U_{n-2} \right)
\end{aligned}
$$

Simple Illustration

$$
\begin{aligned}
&= 2 \cdot 2|\mathcal{B}_{n-1}|\mathbb{E}_{n-1}\left(\frac{3}{2}\right)^{U_{n-2}} \\
&= 2 \cdot 2|\mathcal{B}_{n-1}|\frac{2|\mathcal{B}_{n-2}|}{|\mathcal{B}_{n-1}|}\mathbb{E}_{n-2}\left(2^{U_{n-2}}\left(\frac{3}{2}\right)^{U_{n-2}}\right) \\
&= 2^2 \cdot 2!\,|\mathcal{B}_{n-2}|\mathbb{E}_{n-2}3^{U_{n-2}}.
\end{aligned}
$$

Iterating n times,

$$
\begin{aligned}
|\mathcal{B}_n| &= 2^{n-1}(n-1)!|\mathcal{B}_1|\mathbb{E}_1 n^{U_1} \\
&= 2^n n!,
\end{aligned}
$$

where the final equality holds because $|\mathcal{B}_1| = 2$ and $U_1 \equiv 1$.

Corners in Type-B Permutation Tableaux

Theorem (Hitczenko, L.)

For type-B permutation tableaux of size n we have

$$\mathbb{E}_n C_n = \frac{4n+7}{24} - \frac{1}{2n}.$$

Proof.

- $\mathbb{E}_n \left(\sum_{k=1}^{n-1} I_{M_k=S, M_{k+1}=W} \right) = \sum_{k=1}^{n-1} \mathbb{E}_n \left(I_{M_k=S, M_{k+1}=W} \right)$

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- Apply $\mathbb{E}_n(X) = \frac{2|\mathcal{B}_{n-1}|}{|\mathcal{B}_n|} \mathbb{E}_{n-1}(2^{U_{n-1}(B_{n-1})} X)$

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- Apply $\mathbb{E}_n(X) = \frac{2|\mathcal{B}_{n-1}|}{|\mathcal{B}_n|} \mathbb{E}_{n-1}(2^{U_{n-1}(B_{n-1})} X)$
- Use the fact that $\mathbb{E}_k(I_{M_k=S} | U_{k-1}) = \frac{1}{2^{U_{k-1}+1}}$