The degree sequence of a random graph

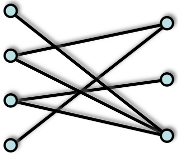
& asymptotic enumeration of regular graphs

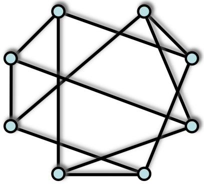
Nick Wormald
Monash University

Including recent work with Anita Liebenau

Some combinatorial objects

bipartite graph





3-regular graph

binary matrix

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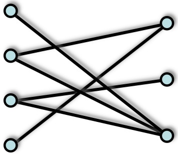

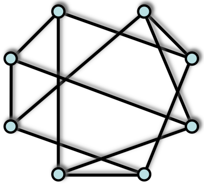

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How many are there (with given vertex degrees / row and column sums)?

Notation:

$n =$ number of vertices, $\quad \mathbf{d} = (d_1, \ldots, d_n) =$ degree sequence

Asymptotics as $n \to \infty$

Read (58) 3-regular graphs:

$$g_3(n) \sim \frac{(3n)!e^{-2}}{(3n/2)!288^{n/2}}.$$

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bipartite d-regular graphs (d fixed).

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graphs with max degree $\Delta = O(1)$.

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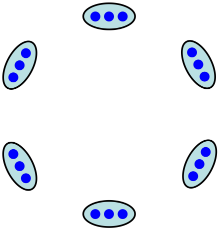

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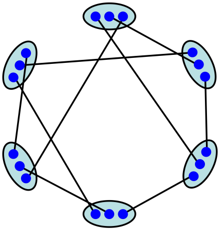

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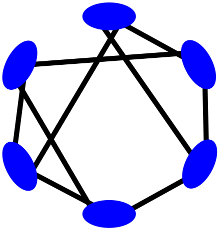

The formula

Number of graphs with degrees $d_1, \ldots, d_n$ is

$$g(d_1, \ldots, d_n) = \frac{|\Phi|\mathbb{P}(\mathrm{SIMPLE})}{\prod d_i!}$$

where

Φ is the set of all pairings, so $|\Phi| = \frac{M_1!}{(M_1/2)!2^{M_1/2}}$ $(M_1 = \sum d_i)$,

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If $\Delta \ = \max_i\{d_i\}$ is bounded, method of moments gives

$$\mathbb{P}(\text{SIMPLE}) \sim e^{-M_2/2M_1 - M_2^2/4M_1^2}$$

where $M_1 = \sum_{i=1}^n d_i, \ \ M_2 = \sum_{i=1}^n d_i(d_i - 1)$.

Reaching for higher degrees

Theorem [McKay 85]

If $\Delta = o(M_1^{1/4})$ then

$$\mathbb{P}(\mathrm{SIMPLE}) \sim e^{-M_2/2M_1 - M_2^2/4M_1^2}.$$

where $\Delta = \max_i\{d_i\}$, $M_k = \sum_{i=1}^{n} [d_i]_k$.

(Same formula as for Δ bounded.)

McKay's method: switchings

$\mathcal{C}_i$: set of pairings with i double edges. (Loops work similarly.)

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Set-up:

$$\frac{1}{\mathbb{P}(\mathcal{C}_0)} = \frac{|\Phi|}{|\mathcal{C}_0|} \approx \sum_{i=0}^{i_{\max}} \frac{|\mathcal{C}_i|}{|\mathcal{C}_0|}$$

where $i_{\max}$ is sufficiently large.

Compute $\dfrac{|\mathcal{C}_i|}{|\mathcal{C}_0|}$ as a telescoping product of $\dfrac{|\mathcal{C}_i|}{|\mathcal{C}_{i-1}|}$.

Switchings ctd

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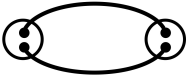

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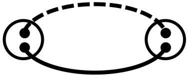

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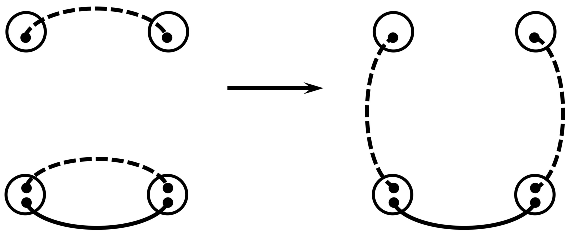

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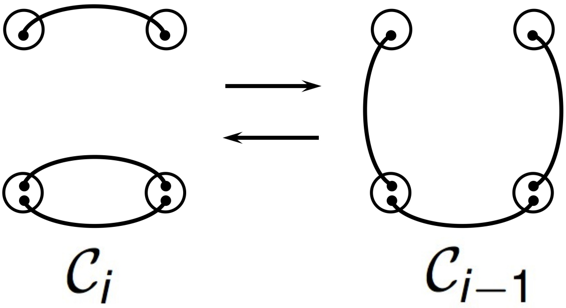

Even higher degree

Theorem [McKay & W 91]

If $\Delta = o(M_1^{1/3})$

$$\mathbb{P}(\text{SIMPLE}) = \exp\Big(-\frac{M_2}{2M_1} - \frac{M_2^2}{4M_1^2} - \frac{M_2^2 M_3}{2M_1^4} + \frac{M_2^4}{4M_1^5} + \frac{M_3^2}{6M_1^3} + O\Big(\frac{\Delta^3}{M_1}\Big)\Big)$$

This covers d-regular for $d = o(\sqrt{n})$.

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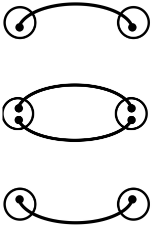

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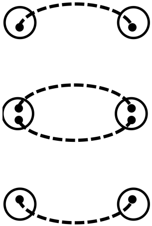

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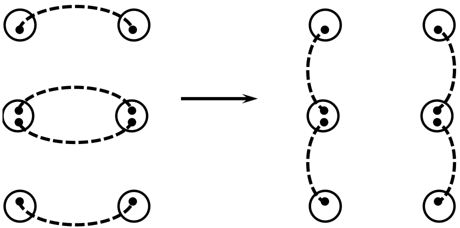

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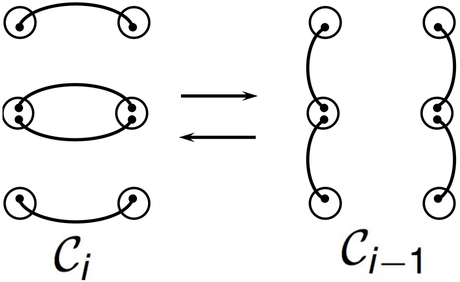

Other sparse cases

Theorem [Janson 09, 14]

For $M_1 \to \infty$ with $M_2 = O(M_1)$,
$\mathbb{P}(\text{SIMPLE}) = o(1) + \exp\left(-\frac{1}{2}\sum \lambda_{ii} - \sum_{i<j}\left(\lambda_{ij} - \log(1+\lambda_{ij})\right)\right)$
where $\lambda_{ij} = \sqrt{d_i(d_i-1)d_j(d_j-1)}/(2M_1)$.

Theorem [Gao & W 16]

Let $\mathbf{d} = (d_1, \ldots, d_n)$ be a function of n. For an appropriate function $\xi(\mathbf{d})$, if $\xi(\mathbf{d}) = o(1)$, then $\mathbb{P}(\text{SIMPLE}) \sim$
$\exp\left(-\frac{M_1}{2} + \frac{M_2}{2M_1} - \frac{M_3}{3M_1^2} + \frac{3}{4} + \sum_{i<j}\log(1 + d_i d_j/M_1)\right)$.

The latter is the first case that applies to power law degree sequences with $\frac{5}{2} < \gamma < 3$, and to sparse degree sequences with $\Delta \gg \sqrt{n}$ (but not d-regular with $d > \sqrt{n}$).

Very high degrees

McKay & W (90): formula for av. degree at least $cn/\log n$, provided variation in degrees is not too large. Method uses:

$$
\begin{aligned}
g(\mathbf{d}) &= [x_1^d \cdots x_n^d] \prod_{i<j} (1 + x_i x_j) \\
&= \frac{1}{(2\pi i)^n} \oint \cdots \oint \frac{\prod_{j<k}(1 + z_j z_k)}{z_1^{d+1} z_2^{d+1} \cdots z_n^{d+1}} \, dz_1 dz_2 \cdots dz_n
\end{aligned}
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where the paths of integration are simple closed contours enclosing the origin.

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Barvinok & Hartigan (12): Related integral approach, much wider range of degrees, but formula not explicit.

The formula for typical degrees

$$
\begin{aligned}
d &:= M_1/n && \text{(average degree)}\\
\lambda &:= d/(n-1) && \text{(edge density)}\\
\gamma_2 &:= \textstyle\sum_{j=1}^{n}(d_j-d)^2/n^2 && \text{(scaled variance of degrees)}
\end{aligned}
$$

Theorem [McKay & W 91]

Suppose that $dn \to \infty$ and $d \le n/2$, and either

- $\Delta = o(n^{1/3}d^{1/3})$ (very sparse case), or
- $d > \frac{2}{3}n/\log n$ (very dense case).

Suppose also $\max\{|d_j - d|\} = o(\max\{d^{1/2+\epsilon}, n^{1/8}\})$. Then

$$g(\mathbf{d}) \sim \exp\left(\frac{1}{4} - \frac{\gamma_2^2}{4\lambda^2(1-\lambda)^2}\right)\sqrt{2}\left(\lambda^\lambda(1-\lambda)^{1-\lambda}\right)^{\binom{n}{2}}\prod_{j=1}^{n}\binom{n-1}{d_j}.$$

Asymptotic Enumeration Conjecture (McKay & W 90)

The same formula holds for all $d \leq n/2$ such that $dn \to \infty$, i.e. the number of edges tends to infinity.

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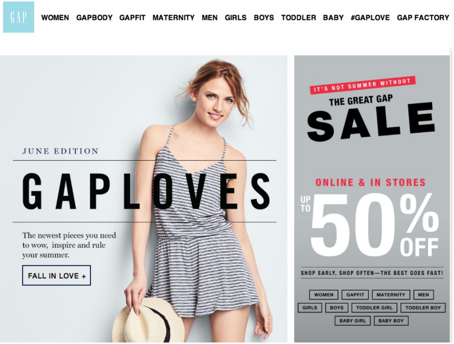

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The Gap (Sydney)

... it has gained infamy for suicides.

The formula again

$$\begin{aligned} d &:= M_1/n && \text{(average degree)} \\ \lambda &:= d/(n-1) && \text{(edge density)} \end{aligned}$$

$$g(\mathbf{d}) \sim \exp\left(\frac{1}{4} - \frac{\gamma_2^2}{4\lambda^2(1-\lambda)^2}\right)\sqrt{2}(\lambda^\lambda(1-\lambda)^{1-\lambda})^{\binom{n}{2}} \prod_{j=1}^{n} \binom{n-1}{d_j}.$$

Set $m = M_1/2$.

$$\frac{\sqrt{2}(\lambda^\lambda(1-\lambda)^{1-\lambda})^{\binom{n}{2}} \prod_{j=1}^{n} \binom{n-1}{d_j}}{\binom{\binom{n}{2}}{m}} \sim \frac{\prod_{j=1}^{n} \binom{n-1}{d_j}(\lambda^\lambda(1-\lambda)^{1-\lambda})^{n-1}}{(\lambda^\lambda(1-\lambda)^{1-\lambda})^{n(n-1)}\binom{n(n-1)}{2m}}$$

i.e.

$$\frac{\sqrt{2}(\lambda^\lambda(1-\lambda)^{1-\lambda})^{\binom{n}{2}} \prod_{j=1}^{n} \binom{n-1}{d_j}}{|\mathcal{G}(n,m)|} \sim \frac{\mathbb{P}(\mathbf{d}) \text{ in indept binomials}}{\mathbb{P}(\mathrm{Bin}(n(n-1),\lambda) = 2m)}$$

The degree sequence of $\mathcal{G}(n,p)$

- Two sequences of probability spaces, A_n and B_n, are asymptotically quite equivalent (a.q.e.) if
$$\Pr_{A_n}(H_n) \sim \Pr_{B_n}(H_n)$$
for all events H_n having probability at least $n^{-O(1)}$.
- $\mathcal{B}_p(n)$ – random sequence of n independent binomial variables $\mathrm{Bin}(n-1,p)$
- $\mathcal{D}(G)$ – degree sequence of a graph G

Binomial Approximation Conjecture (McKay & W 95)

For integer m let $p = \frac{m}{\binom{n}{2}}$ and assume that $p(1-p) \gg \frac{\log n}{n^2}$. Then

- $\mathcal{D}(\mathcal{G}(n,m))$ and $\mathcal{B}_p(n)\big|_{\Sigma=2m}$ are a.q.e.
- $\mathcal{D}(\mathcal{G}(n,p))$ and $\mathcal{B}_{\hat{p}}(n)\big|_{\Sigma \text{ is even}}$ are a.q.e., where $\hat{p}$ is a truncated normal variable, tightly concentrated near p

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We showed that this conjecture is true for any density where the Asymptotic Enumeration Conjecture holds. So it has the same gap.

The degree sequence of the random graph $\mathcal{G}(n, p)$

Many results derived in 70s – 90s such as

- Distribution of number of vertices of degree k (Bollobás 80s, Barbour, Karoński and Ruciński 89, Barbour, Holst and Janson 92).
- Various results about d_m, the mth largest degree (Bollobás, Pałka). Examples: distribution when m is fixed, when is d_1 determined asymptotically, when is there a.a.s. a unique vertex of min or max degree, ...
- With probability $1 - O(n^{-K})$ all degrees d_i satisfy $|d_i - np| = O(\sqrt{np \log n})$ — unless $p(1-p)$ is *very* small.

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New approach for enumeration by degree sequence

Compare numbers of graphs with different degree sequences:

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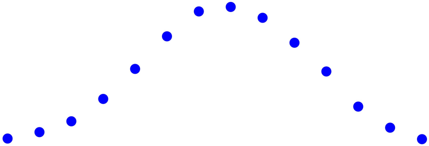

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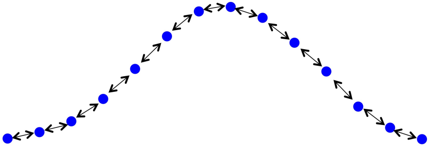

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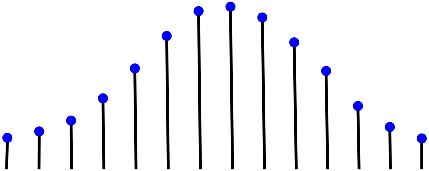

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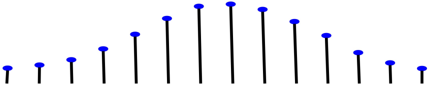

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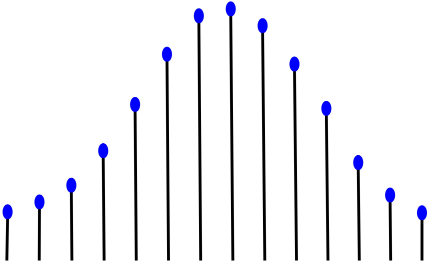

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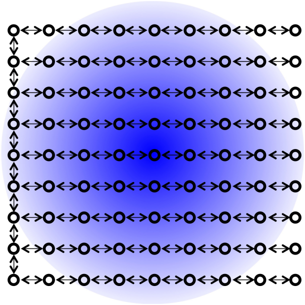

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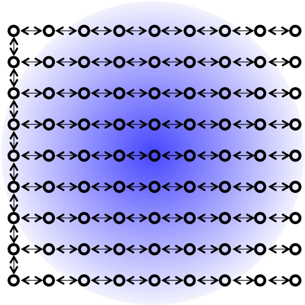

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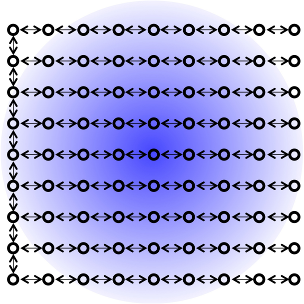
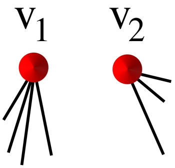


Length of telescoping products required is $O(n\sqrt{d})$.
$\longrightarrow$ desired accuracy in each ratio is $o\big(1/(n\sqrt{d})\big)$

Estimating the ratio by 'degree switching'

Choose a random edge incident with v_1 and move it to v_2.



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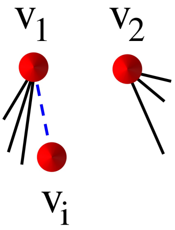

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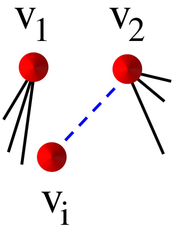

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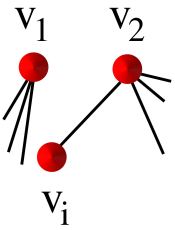

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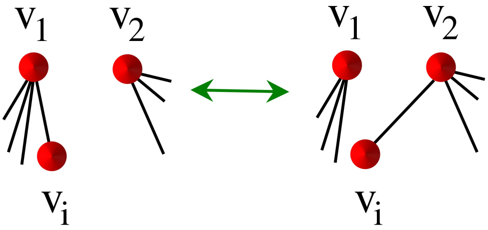

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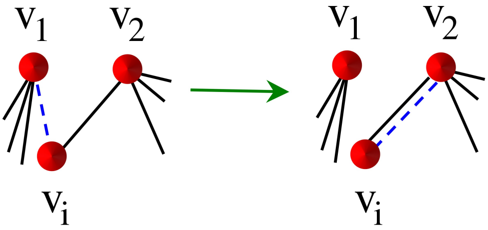
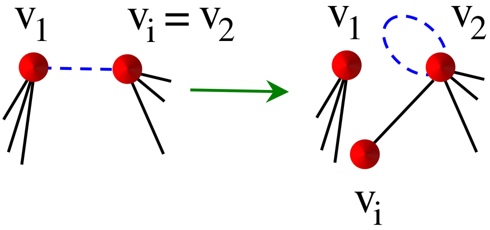


A bad choice giving a multiple edge

Estimating the ratio by 'degree switching'

Choose a random edge incident with v_1 and move it to v_2.



A bad choice giving a loop

Estimating the ratio by 'degree switching'

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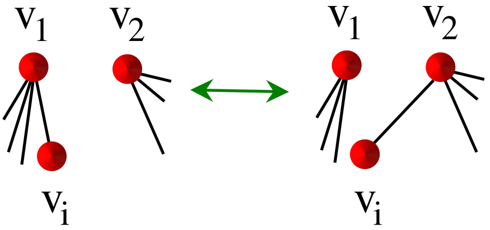

$$\mathbf{d} - \mathbf{e}_2 \qquad \longleftrightarrow \qquad \mathbf{d} - \mathbf{e}_1$$

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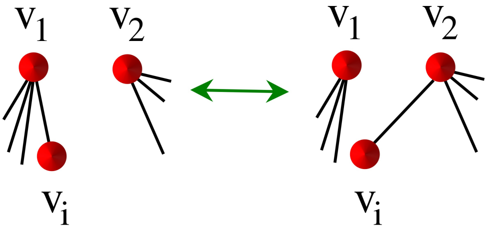

$$\mathbf{d} - \mathbf{e}_2 \qquad \longleftrightarrow \qquad \mathbf{d} - \mathbf{e}_1$$

$\#$ ways $\longrightarrow$: $\quad d_1 g(\mathbf{d} - \mathbf{e}_2)\big(1 - \mathbb{P}(\mathsf{bad}_{12}(\mathbf{d} - \mathbf{e}_2))\big)$

$\#$ ways $\longleftarrow$: $\quad d_2 g(\mathbf{d} - \mathbf{e}_1)\big(1 - \mathbb{P}(\mathsf{bad}_{21}(\mathbf{d} - \mathbf{e}_1))\big)$

Estimating the ratio by 'degree switching'

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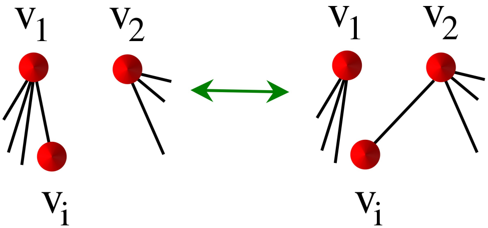

$$\mathbf{d}-\mathbf{e}_2 \qquad \longleftrightarrow \qquad \mathbf{d}-\mathbf{e}_1$$

$$\left.\begin{array}{ll} \#\text{ ways } \longrightarrow: & d_1 g(\mathbf{d}-\mathbf{e}_2)\big(1-\mathbb{P}(\mathsf{bad}_{12}(\mathbf{d}-\mathbf{e}_2))\big) \\ \#\text{ ways } \longleftarrow: & d_2 g(\mathbf{d}-\mathbf{e}_1)\big(1-\mathbb{P}(\mathsf{bad}_{21}(\mathbf{d}-\mathbf{e}_1))\big) \end{array}\right\}\text{ equal}$$

Degree switching ctd

$$\left.\begin{array}{ll} \#\ \text{ways} \longrightarrow: & d_1 g(\mathbf{d}-\mathbf{e}_2)\big(1-\mathbb{P}(\mathsf{bad}_{12}(\mathbf{d}-\mathbf{e}_2))\big) \\ \#\ \text{ways} \longleftarrow: & d_2 g(\mathbf{d}-\mathbf{e}_1)\big(1-\mathbb{P}(\mathsf{bad}_{21}(\mathbf{d}-\mathbf{e}_1))\big) \end{array}\right\}\ \text{equal}$$

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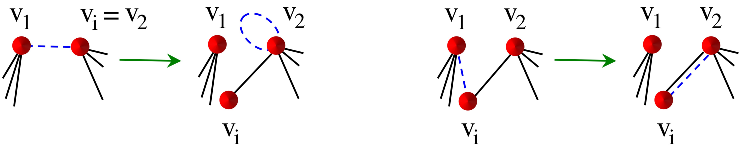
Thus

$$R_{12}(\mathbf{d}) = \frac{d_1\big(1-\mathbb{P}(\mathsf{bad}_{12}(\mathbf{d}-\mathbf{e}_2))\big)}{d_2\big(1-\mathbb{P}(\mathsf{bad}_{21}(\mathbf{d}-\mathbf{e}_1))\big)}$$

Estimating $\mathbb{P}(\mathsf{bad}_{12}(\mathbf{d}))$

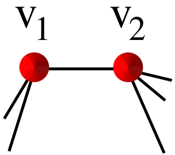


$$\mathbb{P}(\mathsf{bad}_{12}(\mathbf{d})) = \frac{1}{d_1} P_{12}(\mathbf{d}) + \frac{1}{d_1} \sum_{i \geq 3} P_{1i,i2}(\mathbf{d})$$

where

$$\begin{aligned} P_{12}(\mathbf{d}) &= \mathbb{P}(v_1 v_2 \in E(G)), \\ P_{1i,i2}(\mathbf{d}) &= \mathbb{P}(v_1 v_i, v_i v_2 \in E(G)) \end{aligned}$$

Estimating $P_{12}(\mathbf{d})$ - first attempt



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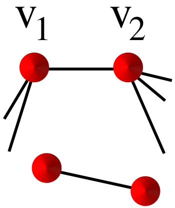

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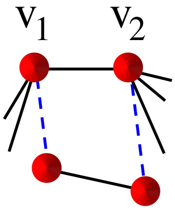

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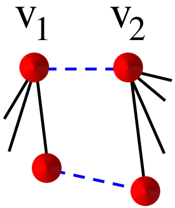

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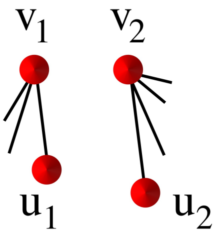

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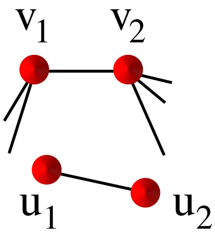

Result of estimating $P_{12}(\mathbf{d})$

- Switchings are prevented by the presence of unwanted edges. Their probabilities can be estimated by secondary switchings, and so on.
- Switchings k-deep indexed by graphs with up to $3k$ edges.
- Similar computation for $P_{1i,i2}$.
- Maple computations use up to 2Gb memory for $k = 6$.
- $R_{12}(\mathbf{d})$ computed to verify both conjectures (binomial-type distribution, and asymptotic formula) for $d = o(n^{4/5})$

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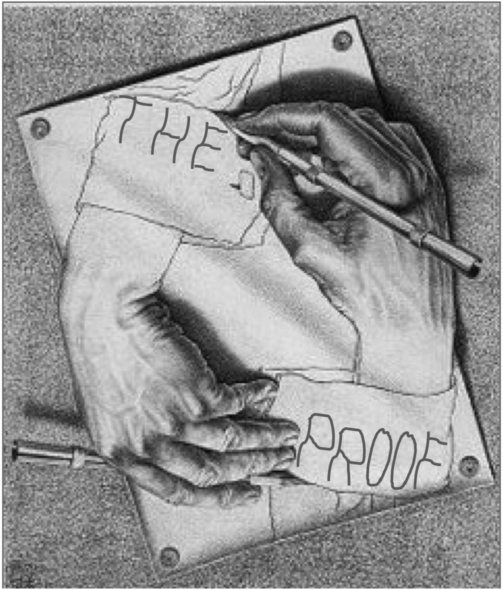
The gap is now much smaller.  But ...

New results

The gap is now completely filled:

Theorem [Liebanau & W 16$^+$]

- Binomial approximation conjecture holds. New argument covers $d = o(n/\sqrt{\log n})$.
- Conjectured formula for graphs with given degrees holds. In particular, for regular graphs.


THE
PROOF

Simple observation

Let $g_{ab}(\mathbf{d})$ be the number of graphs with degree sequence $\mathbf{d}$ and containing the edge $v_a v_b$. Thus

$$g_{ab}(\mathbf{d}) = g(\mathbf{d}) P_{ab}(\mathbf{d}).$$

Removing the edge shows that

$$g_{ab}(\mathbf{d}) = g(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b}) - g_{ab}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})$$

and hence we have a simple observation:

$$g_{ab}(\mathbf{d}) \quad = \quad g(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})\big(1 - P_{ab}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})\big).$$

Second approach for $P_{12}(\mathbf{d})$

$$d_1 = \mathbb{E}\deg(\mathrm{v}_1) = \sum_{\mathrm{i}=2}^{\mathrm{n}} \mathrm{P}_{1\mathrm{i}}(\mathbf{d})$$

(by linearity of expectation)

$$= P_{12}(\mathbf{d}) \sum_{i=2}^{n} \frac{P_{1i}(\mathbf{d})}{P_{12}(\mathbf{d})} = P_{12}(\mathbf{d}) \sum_{i=2}^{n} \frac{g_{1i}(\mathbf{d})}{g_{12}(\mathbf{d})}$$

$$\begin{aligned} &= P_{12}(\mathbf{d}) \sum_{i=2}^{n} \frac{g(\mathbf{d}-\mathbf{e_1}-\mathbf{e_i})\big(1-P_{1i}(\mathbf{d}-\mathbf{e_1}-\mathbf{e_i})\big)}{g(\mathbf{d}-\mathbf{e_1}-\mathbf{e_2})\big(1-P_{12}(\mathbf{d}-\mathbf{e_1}-\mathbf{e_2})\big)} \\ d_1 &= P_{12}(\mathbf{d}) \sum_{i=2}^{n} R_{i2}(\mathbf{d}-\mathbf{e}_1) \frac{1-P_{1i}(\mathbf{d}-\mathbf{e_1}-\mathbf{e_i})}{1-P_{12}(\mathbf{d}-\mathbf{e_1}-\mathbf{e_2})} \end{aligned}$$

Second approach for $P_{1i,i2}(\mathbf{d})$

The same equation from removing an edge

$$g_{ab}(\mathbf{d}) = g(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b}) - g_{ab}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})$$

iterates to give our second simple observation:

$$\begin{aligned}
g_{ab}(\mathbf{d}) &= g(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b}) - g_{ab}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b}) \\
&= g(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b}) - g(\mathbf{d} - 2\mathbf{e_a} - 2\mathbf{e_b}) + g_{ab}(\mathbf{d} - 2\mathbf{e_a} - 2\mathbf{e_b}) \\
&. \\
&. \\
&. \\
&= \sum_{k \geq 1} (-1)^{k-1} g(\mathbf{d} - k\mathbf{e_a} - k\mathbf{e_b})
\end{aligned}$$

Computation for $P_{1i,i2}(\mathbf{d})$ (ctd)

That equation again:

$$g_{ab}(\mathbf{d}) = \sum_{k \geq 1} (-1)^{k-1} g(\mathbf{d} - k\mathbf{e_a} - k\mathbf{e_b})$$

Similarly edge $1i$ fixed and operating on $i2$:

$$g_{1i,i2}(\mathbf{d}) = \sum_{k \geq 1} (-1)^{k-1} g_{1i}(\mathbf{d} - k\mathbf{e_i} - k\mathbf{e_2})$$

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$$g_{1i,i2}(\mathbf{d}) = \sum_{k \geq 1} (-1)^{k-1} g_{1i}(\mathbf{d} - k\mathbf{e_i} - k\mathbf{e_2})$$

$$\frac{g_{1i,i2}(\mathbf{d})}{g(\mathbf{d})} = \sum_{k \geq 1} (-1)^{k-1} \frac{P_{1i}(\mathbf{d} - k\mathbf{e_i} - k\mathbf{e_2}) g(\mathbf{d} - k\mathbf{e_i} - k\mathbf{e_2})}{g(\mathbf{d})}$$

$$P_{1i,i2}(\mathbf{d}) = \sum_{k \geq 1} (-1)^{k-1} P_{1i}(\mathbf{d} - k\mathbf{e_i} - k\mathbf{e_2}) \frac{g(\mathbf{d} - k\mathbf{e_i} - k\mathbf{e_2})}{g(\mathbf{d})}$$

Computation for $P_{1i,i2}(\mathbf{d})$ (ctd 2)

That equation again:

$$P_{1i,i2}(\mathbf{d}) = \sum_{k\geq 1}(-1)^{k-1}P_{1i}(\mathbf{d}-k\mathbf{e_i}-k\mathbf{e_2})\frac{g(\mathbf{d}-k\mathbf{e_i}-k\mathbf{e_2})}{g(\mathbf{d})}$$

Our third simple observation:

$$\frac{g(\mathbf{d}-k\mathbf{e_i}-k\mathbf{e_2})}{g(\mathbf{d}-(k-1)\mathbf{e_i}-(k-1)\mathbf{e_2})} = \frac{P_{i2}(\mathbf{d}-(k-1)\mathbf{e_i}-(k-1)\mathbf{e_2})}{1-P_{i2}(\mathbf{d}-k\mathbf{e_i}-k\mathbf{e_2})}$$

finally gives

$$P_{1i,i2}(\mathbf{d}) = \mathcal{F}(P_{1i}, P_{i2}, P_{1i}(\mathbf{d}-\mathbf{e_i}-\mathbf{e_2}), \ldots)$$

The resulting equations for P_{ab} and R_{ab}

$$
\begin{aligned}
P_{ab}(\mathbf{d}) &= d_a \left(\sum_{i \neq a} R_{ib}(\mathbf{d} - \mathbf{e}_a) \frac{1 - P_{ai}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})}{1 - P_{ab}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})} \right)^{-1} \\
R_{ab}(\mathbf{d}) &= \frac{d_a \big(1 - \mathbb{P}(\mathsf{bad}_{ab}(\mathbf{d} - \mathbf{e}_b))\big)}{d_b \big(1 - \mathbb{P}(\mathsf{bad}_{ba}(\mathbf{d} - \mathbf{e}_a))\big)}
\end{aligned}
$$

where $\mathbb{P}(\mathsf{bad}_{ab})$, $\mathbb{P}(\mathsf{bad}_{ba})$ are functions of various P_{ij}.

RHS's can be viewed as an operator on the set of P_{ab} and R_{ab} for which we desire an appropriate fixed point.

Initial step of recursive computation

With $d = (n-1)p$, assume $d = o(n/\sqrt{\log n})$.
We can assume things that hold 'asymptotically quite surely' for say $d > n^{1/3}$:

- $|d_i - d| = O(\sqrt{d \log n})$ for all i;
- $\sum |d_i - d|^2 = dn + O(d^{3/4} n)$

First step: by ordinary switchings it's easy to see

$$P_{12}(\mathbf{d}) = O(d/n)$$

Recursive computation

$$P_{ab}(\mathbf{d}) = d_a \left(\sum_{i \neq a} R_{ib}(\mathbf{d} - \mathbf{e}_a) \frac{1 - P_{ai}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})}{1 - P_{ab}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})} \right)^{-1}$$

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$$P_{12}(\mathbf{d}) = O\left(\frac{d}{n}\right)$$

Recursive computation

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$$P_{12}(\mathbf{d}) = O\left(\frac{d}{n}\right)$$

$$\Downarrow$$

$$R_{12}(\mathbf{d}) = \frac{d_1}{d_2} + O\left(\frac{d}{n}\right)$$

Recursive computation

$$
\begin{aligned}
P_{ab}(\mathbf{d}) &= d_a \left(\sum_{i \neq a} R_{ib}(\mathbf{d} - \mathbf{e}_a) \frac{1 - P_{ai}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})}{1 - P_{ab}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})} \right)^{-1} \\
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\end{aligned}
$$

$$P_{12}(\mathbf{d}) = \frac{d_1 d_2}{dn} + O\left(\frac{d^2}{n^2}\right)$$

$$\Uparrow$$

$$R_{12}(\mathbf{d}) = \frac{d_1}{d_2} + O\left(\frac{d}{n}\right)$$

Recursive computation

$$P_{ab}(\mathbf{d}) = d_a\left(\sum_{i\neq a} R_{ib}(\mathbf{d}-\mathbf{e}_a)\frac{1-P_{ai}(\mathbf{d}-\mathbf{e_a}-\mathbf{e_b})}{1-P_{ab}(\mathbf{d}-\mathbf{e_a}-\mathbf{e_b})}\right)^{-1}$$

$$R_{ab}(\mathbf{d}) = \frac{d_a\big(1-\mathbb{P}(\mathsf{bad}_{ab}(\mathbf{d}-\mathbf{e}_b))\big)}{d_b\big(1-\mathbb{P}(\mathsf{bad}_{ba}(\mathbf{d}-\mathbf{e}_a))\big)}$$

$$P_{12}(\mathbf{d}) = \frac{d_1 d_2}{dn} + O\left(\frac{d^2}{n^2}\right)$$

$$\Downarrow$$

$$R_{12}(\mathbf{d}) = \frac{d_1(n-d_2)}{d_2(n-d_1)}\left(1+O\left(\frac{1}{n}\right)\right)$$

Recursive computation

$$
\begin{aligned}
P_{ab}(\mathbf{d}) &= d_a \left(\sum_{i \neq a} R_{ib}(\mathbf{d} - \mathbf{e}_a) \frac{1 - P_{ai}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})}{1 - P_{ab}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})} \right)^{-1} \\
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\end{aligned}
$$

$$
P_{12}(\mathbf{d}) = \frac{d_1 d_2}{dn} \left(1 - \frac{(d - d_1)(d - d_2)}{dn} \right) + O\left(\frac{d^3}{n^3} \right)
$$

$$
\Lsh\!\Uparrow \quad \Uparrow
$$

$$
R_{12}(\mathbf{d}) = \frac{d_1 (n - d_2)}{d_2 (n - d_1)} + O\left(\frac{1}{n} \right)
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Recursive computation

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\end{aligned}
$$

$$
P_{12}(\mathbf{d}) = \frac{d_1 d_2}{dn}\left(1-\frac{(d-d_1)(d-d_2)}{dn}\right) + O\left(\frac{d^3}{n^3}\right)
$$

$$\Downarrow$$

$$
R_{12}(\mathbf{d}) = \frac{d_1(n-d_2)}{d_2(n-d_1)} + O\left(\frac{\sqrt{d\log n}}{n^2}\right)
$$

Recursive computation

$$P_{ab}(\mathbf{d}) \;=\; d_a\left(\sum_{i\neq a} R_{ib}(\mathbf{d}-\mathbf{e}_a)\frac{1-P_{ai}(\mathbf{d}-\mathbf{e_a}-\mathbf{e_b})}{1-P_{ab}(\mathbf{d}-\mathbf{e_a}-\mathbf{e_b})}\right)^{-1}$$

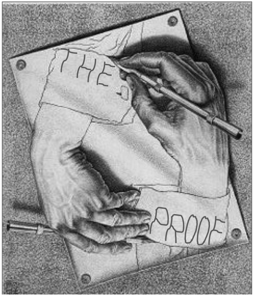
$$R_{ab}(\mathbf{d}) \;=\; \frac{d_a\big(1-\mathbb{P}(\mathsf{bad}_{ab}(\mathbf{d}-\mathbf{e}_b))\big)}{d_b\big(1-\mathbb{P}(\mathsf{bad}_{ba}(\mathbf{d}-\mathbf{e}_a))\big)}$$

$$P_{12}(\mathbf{d}) = \frac{d_1 d_2}{dn}\left(1-\frac{(d-d_1)(d-d_2)}{dn}\left(1+\frac{dd_1+dd_2-d_1d_2}{dn}\right)\right)+O\left(\frac{d^4}{n^4}\right)$$

$$R_{12}(\mathbf{d}) = \frac{d_1(n-d_2)}{d_2(n-d_1)} + O\left(\frac{\sqrt{d\log n}}{n^2}\right)$$

Recursive computation

$$
\begin{aligned}
P_{ab}(\mathbf{d}) &= d_a \left(\sum_{i \neq a} R_{ib}(\mathbf{d} - \mathbf{e}_a) \frac{1 - P_{ai}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})}{1 - P_{ab}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})} \right)^{-1} \\
R_{ab}(\mathbf{d}) &= \frac{d_a \big(1 - \mathbb{P}(\mathsf{bad}_{ab}(\mathbf{d} - \mathbf{e}_b))\big)}{d_b \big(1 - \mathbb{P}(\mathsf{bad}_{ba}(\mathbf{d} - \mathbf{e}_a))\big)}
\end{aligned}
$$



Recursive computation

$$P_{ab}(\mathbf{d}) = d_a \left(\sum_{i \neq a} R_{ib}(\mathbf{d} - \mathbf{e}_a) \frac{1 - P_{ai}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})}{1 - P_{ab}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})} \right)^{-1}$$

$$R_{ab}(\mathbf{d}) = \frac{d_a\big(1 - \mathbb{P}(\mathsf{bad}_{ab}(\mathbf{d} - \mathbf{e}_b))\big)}{d_b\big(1 - \mathbb{P}(\mathsf{bad}_{ba}(\mathbf{d} - \mathbf{e}_a))\big)}$$

$$P_{12}(\mathbf{d}) = \frac{d_1 d_2}{dn}\left(1 - \frac{(d-d_1)(d-d_2)}{dn}\left(1 + T + T^2\right)\right) + O\left(\frac{d^5}{n^5}\right)$$

$(T = (dd_1 + dd_2 - d_1 d_2)/dn)$

$$R_{12}(\mathbf{d}) = \frac{d_1(n-d_2)}{d_2(n-d_1)} + O\left(\frac{\sqrt{d \log n}}{n^2}\right)$$

Recursive computation

$$
\begin{aligned}
P_{ab}(\mathbf{d}) &= d_a \left(\sum_{i \neq a} R_{ib}(\mathbf{d} - \mathbf{e}_a) \frac{1 - P_{ai}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})}{1 - P_{ab}(\mathbf{d} - \mathbf{e_a} - \mathbf{e_b})} \right)^{-1} \\
R_{ab}(\mathbf{d}) &= \frac{d_a \big(1 - \mathbb{P}(\mathsf{bad}_{ab}(\mathbf{d} - \mathbf{e}_b))\big)}{d_b \big(1 - \mathbb{P}(\mathsf{bad}_{ba}(\mathbf{d} - \mathbf{e}_a))\big)}
\end{aligned}
$$

$$P_{12}(\mathbf{d}) = \frac{d_1 d_2}{dn}\left(1 - \frac{(d-d_1)(d-d_2)}{dn}\left(1 + T + T^2 + T^3\right)\right) + O\left(\frac{d^6}{n^6}\right)$$

$(T = (dd_1 + dd_2 - d_1 d_2)/dn)$

$$R_{12}(\mathbf{d}) = \frac{d_1(n-d_2)}{d_2(n-d_1)} + O\left(\frac{\sqrt{d \log n}}{n^2}\right)$$

Recursive computation

$$P_{ab}(\mathbf{d}) \quad = \quad d_a\left(\sum_{i\neq a} R_{ib}(\mathbf{d}-\mathbf{e}_a)\frac{1-P_{ai}(\mathbf{d}-\mathbf{e_a}-\mathbf{e_b})}{1-P_{ab}(\mathbf{d}-\mathbf{e_a}-\mathbf{e_b})}\right)^{-1}$$

$$R_{ab}(\mathbf{d}) \quad = \quad \frac{d_a\big(1-\mathbb{P}(\mathsf{bad}_{ab}(\mathbf{d}-\mathbf{e}_b))\big)}{d_b\big(1-\mathbb{P}(\mathsf{bad}_{ba}(\mathbf{d}-\mathbf{e}_a))\big)}$$

$$P_{12}(\mathbf{d}) = \frac{d_1 d_2(n-d)}{n(dn-d_1d-d_2d+d_1d_2)} + O\Big(\frac{\sqrt{d\log n}}{n^2}\Big)$$

[Both these relations are now easily verified]

$$R_{12}(\mathbf{d}) = \frac{d_1(n-d_2)}{d_2(n-d_1)} + O\Big(\frac{\sqrt{d\log n}}{n^2}\Big)$$

Results

This proves the binomial approximation conjecture.

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Continuing the same process to get more accuracy gives the asymptotic formula conjecture in the gap (and with a wider range of degrees):

Theorem [Liebenau & W 16^+]

Let $d = \frac{1}{n}\sum d_i$ and $\lambda = d/(n-1)$. Suppose that $\sum d_i$ is even, $\log^3 n < d = o(n/\sqrt{\log n})$, $\max\{|d_j - d|\} = o(d^{3/5})$. Then

$$g(\mathbf{d}) \sim \exp\left(\frac{1}{4} - \frac{\gamma_2^2}{4\lambda^2(1-\lambda)^2}\right)\sqrt{2}(\lambda^\lambda(1-\lambda)^{1-\lambda})^{\binom{n}{2}} \prod_{j=1}^{n} \binom{n-1}{d_j}.$$

d-regular case

$$g(d) \sim e^{\frac{1}{4}}\sqrt{2}\left(\frac{d^d(n-1-d)^{n-1-d}}{(n-1)^{n-1}}\right)^{n/2}\binom{n-1}{d}^n.$$

McKay and Isaev have announced another proof of this (mid June 2016) for $d \gg n^{3/4}$, by analysing the Cauchy integrals using complex martingales.

Digraphs, bipartite graphs (0/1 matrices), hypergraphs

- Almost identical argument gives analogous enumeration results for all these objects.
- This completes the obvious binomial approximation theorems for degree sequences of random digraphs and random bipartite graphs with given numbers of edges. (Very dense cases not yet done for hypergraphs, but Greenhill, Isaev and McKay claim a result for k-uniform.)
- Applies also to restricted versions of hypergraphs (such as linear).

Results with more restricted sparsity

Graphs:

- Read (60)
- Bender & Canfield (78)
- Bollobás (80)
- McKay (85)
- McKay & W (91)

Bipartite graphs:

- O'Neil (69)
- Békessy, Békessy and Komlós (72)
- Bender (74)
- McKay (84)
- McKay & Wang (03)
- Greenhill, McKay & Wang (06)

Hypergraphs:

- Cooper, Frieze, Molloy & Reed (96)
- Dudek, Frieze, Ruciński & Šileikis (13)
- Blinovsky & Greenhill (16)
- Blinovsky & Greenhill (16^+)

What remains

- Work in progress (with Leckey and Liebenau): enumeration of $k \times n$ Latin rectangles.

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- Work in progress (with McKay, since 1990): implications of the binomial model for the degree sequence of a random graph.

- UNSOLVED PROBLEM: number of linear hypergraphs in the dense case ($\sim cn$ edges)