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Asymptotic Expansions in SageMath (Poster)

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We present a toolbox for everyday life in analytic combinatorics, namely the new asymptotic expansion module which is included in the mathematics software system SageMath. The code of this module was contributed by the authors of this poster.

Keywords: Asymptotic expansions, SageMath, software

SageMath [1] is a free and open-source mathematics software system. Since version 6.10, it is shipped with a module for computations with asymptotic expansions [3]; no workaround or loading of a package is needed. Even better, it is automatically tested with each release to guarantee functionality and repreducibility of the results.

The asymptotic expansion module(i) is integrated completely into SageMath's infrastructure and interacts with all of SageMath's other mathematical objects very well. All contributed code and documentation goes through a transparent peer-review process; this ensures that SageMath's quality standards on efficient, readable, and maintainable code are met.

Due to space restrictions, we limit ourselves to the univariate case here. However, the module is designed for multivariate asymptotic expansions as well.

1 Creating an Asymptotic Ring

We use the coefficient ring

```
C = SR.subring(no_variables=True)  # symbolic constants
```

which is a ring of symbolic constants; note that this includes the rationals $\mathbb{Q}$. A univariate asymptotic ring for our calculations is created by

```
A = AsymptoticRing(growth_group='QQ^n * n^QQ * log(n)^QQ',
                   coefficient_ring=C,
                   default_prec=5)
n = A.gen()
```

A typical element of `A` is the asymptotic expansion

$$42\left(\frac{1}{3}\right)^n n^{\frac{5}{2}}\log(n)^2 + O\left(\left(\frac{1}{3}\right)^n n\log(n)^{\frac{1}{2}}\right)$$

as $n \to \infty$.

(i) See `http://doc.sagemath.org/html/en/reference/asymptotic` for the online-documentation.

2 Basic Arithmetic

Beside the very basic arithmetical operations addition, subtraction and multiplications, series expansions are automatically performed for division, the exponential function or the logarithm. For example, typing `log(n + 1)` returns

$$\log(n+1) = \log(n) + n^{-1} - \frac{1}{2}n^{-2} + \frac{1}{3}n^{-3} - \frac{1}{4}n^{-4} + \frac{1}{5}n^{-5} + O\big(n^{-6}\big)\,.$$

More advanced stuff is possible, e.g., `(1 + 1/n)^n` returns

$$\left(1+\frac{1}{n}\right)^n = e - \frac{1}{2}\,en^{-1} + \frac{11}{24}\,en^{-2} - \frac{7}{16}\,en^{-3} + \frac{2447}{5760}\,en^{-4} + O\big(n^{-5}\big)\,.$$

3 Example: Catalan Numbers

There are several possibilities to obtain asymptotics for the Catalan numbers. We use the convenience generator function for $\binom{kn}{n}$ and type

```
binomial_2n_n = asymptotic_expansions.Binomial_kn_over_n(
                    'n', k=2, precision=3)
C_n = binomial_2n_n / (n+1)
```

This results in

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{\sqrt{\pi}}4^n n^{-\frac{3}{2}} - \frac{9}{8\sqrt{\pi}}4^n n^{-\frac{5}{2}} + \frac{145}{128\sqrt{\pi}}4^n n^{-\frac{7}{2}} + O\Big(4^n n^{-\frac{9}{2}}\Big)\,.$$

This could have been achieved by using `factorial()` to build the binomial coefficient manually, as well.

Getting the asymptotic expansions, for example, the harmonic numbers is even easier, since there is a pre-defined generator in SageMath.

4 Singularity Analysis

The Catalan numbers satisfy the generating function

```
def catalan(z):
    return (1 - sqrt(1-4*z)) / (2*z)
```

So, in contrast to the direct calculation of their asymptotic expansion out of the exact formula, we can do a singularity analysis. SageMath assists here as well. We perform

```
C_n = A.coefficients_of_generating_function(
          catalan, singularities=(1/4,), precision=3)
```

to obtain

$$\frac{1}{\sqrt{\pi}}4^n n^{-\frac{3}{2}} - \frac{9}{8\sqrt{\pi}}4^n n^{-\frac{5}{2}} + \frac{145}{128\sqrt{\pi}}4^n n^{-\frac{7}{2}} + O\big(4^n n^{-4}\big)$$

again. We can proceed similarly with the harmonic numbers.

Bootstrapping for finding the dominant singularity is easily possible as well. For example, let us consider longest runs of words over a two letter alphabet, see [2, Example V.4]. The generating function counting runs where one of the two letters has less than n consecutive repetitions is $(1-z^n)/(1-2z+z^{n+1})$. The dominant singularity satisfies the fix-point equation $z = f(z)$ with

```
def f(z):
    return (1 + z^(n+1)) / 2
```

By starting with the approximation $z = \frac{1}{2} + O((\frac{3}{5})^n)$, applying `f` twice yields the known expansion

$$z = \frac{1}{2} + \frac{1}{4}\left(\frac{1}{2}\right)^n + \frac{1}{8}\left(\frac{1}{4}\right)^n n + \frac{1}{8}\left(\frac{1}{4}\right)^n + O\left(\left(\frac{3}{20}\right)^n n^2\right).$$

References

[1] The SageMath Developers, *SageMath Mathematics Software (Version 7.0)*, 2016, `http://www.sagemath.org`.

[2] Philippe Flajolet and Robert Sedgewick, *Analytic combinatorics*, Cambridge University Press, Cambridge, 2009.

[3] Benjamin Hackl, Clemens Heuberger, and Daniel Krenn, *Asymptotic expansions in SageMath*, `http://trac.sagemath.org/17601`, 2015, module in SageMath 6.10.